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# ON THE COMPLETE INTEGRABILITY OF SOME LAX SYSTEMS ON $\mathrm{GL}(n, R) \times \mathrm{GL}(n, R)$

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## INTRODUCTION

Over the past decade there has been a great deal of activity in the solution of nonlinear evolution equations by the Riemann problem method (see [5] and references therein). As is well known, at the basis of all these works is the representation of the equations as a condition of zero curvature, i.e.,

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0. \tag{1}$$

Here, $U$ and $V$ are matrix-valued functions parametrized by the classical fields and $[\cdot, \cdot]$ is the standard commutator. For periodic lattice models, where the (discretized) spatial variable $n$ now takes values in $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$, there is a natural analog of (1). These are the so-called Lax systems [1, 5] and have the general form

$$\frac{dL_n}{dt} = V_{n+1}L_n - L_nV_n, \qquad n \in \mathbf{Z}_N. \tag{2}$$

The matrices $L_n$ above are invertible and define parallel transport from site $n$ of the lattice to site $n+1$ [5]. As can be easily verified, the monodromy matrix $T(L) = L_N \cdots L_1$, $L = (L_1, \ldots, L_N)$ undergoes an isospectral deformation

$$\frac{dT(L)}{dt} = [V_1, T(L)] \tag{3}$$

and hence the eigenvalues of $T(L)$ provide a collection of conserved quantities for (2). In this note, we shall consider a special case of (2) which is related to eigenvalue algorithms and for which additional integrals can be constructed to prove complete integrability (in the sense of Liouville) on generic symplectic leaves. For the convenience of the general reader, we recall that a Poisson

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manifold $P$ is a manifold equipped with a Lie bracket $\{\ ,\ \}$ on $C^\infty(P)$, satisfying the Leibniz identity. Kirillov's theorem [6] says that every such manifold admits a partition into symplectic manifolds, its symplectic leaves. Finally, a Hamiltonian vector field is said to be completely integrable on a symplectic leaf of dimension $2m$ if there exist $m$ integrals $F_1, \ldots, F_m$ functionally independent on an open dense set of the leaf and such that $\{F_i, F_j\} = 0$ $\forall i, j$.

## 1

We now introduce the equations. Let $\mathbf{G}$ be a Lie group whose Lie algebra $\mathbf{g}$ is equipped with a nondegenerate ad-invariant pairing $\langle\cdot,\cdot\rangle$. For $\varphi \in C^\infty(\mathbf{G})$, let $D\varphi$, $D'\varphi$ be its left and right gradients, respectively, thus:

$$(4)\qquad \begin{aligned} \langle D\varphi(g), X\rangle &= \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX}g), \\ \langle D'\varphi(g), X\rangle &= \frac{d}{dt}\Big|_{t=0} \varphi(ge^{tX}). \end{aligned}$$

Since the pairing on $\mathbf{g}$ can be extended to one on $\mathbf{g}_2 = \mathbf{g}\oplus\mathbf{g}$ (Lie algebra direct sum), for a function $\Phi$ on $\mathbf{G}^2 = \mathbf{G}\times\mathbf{G}$, the left and right gradients can be defined as above and we shall write $D\Phi = (D_1\Phi, D_2\Phi)$, $D'\Phi = (D_1'\Phi, D_2'\Phi)$.

In what follows, we take $\mathbf{G}$ to be the group of real, invertible $n\times n$ matrices and consider the standard pairing $\langle A, B\rangle = \operatorname{tr} AB$ on $\mathbf{g}$. We have $\mathbf{g} = \mathbf{k}\oplus\mathbf{l}$ (with the associated projections $\Pi_{\mathbf{k}}$, $\Pi_{\mathbf{l}}$) where $\mathbf{k}$ and $\mathbf{l}$ are the subalgebras of skew-symmetric matrices and lower triangular matrices, respectively. This gives rise to the following classical $r$-matrix [10]

$$(5)\qquad R = \Pi_{\mathbf{k}} - \Pi_{\mathbf{l}}.$$

As a final ingredient for the equations, we take a central function $\psi \in C^\infty(\mathbf{G})$, i.e., $\psi(xyx^{-1}) = \psi(y)$, $x, y \in \mathbf{G}$. We then associate to $\psi$ the function

$$(6)\qquad H_\psi(g_1, g_2) = \psi(T(g)),\ T(g) = g_2 g_1,\ g = (g_1, g_2) \in \mathbf{G}^2.$$

The Lax system we shall consider in this note is given by

$$(7)\qquad \begin{aligned} \dot g_1 &= \tfrac12 R(D_1H_\psi(g))g_1 - \tfrac12 g_1 R(D_2H_\psi(g)), \\ \dot g_2 &= \tfrac12 R(D_2H_\psi(g))g_2 - \tfrac12 g_2 R(D_1H_\psi(g)). \end{aligned}$$

As the reader will see in the section below, these equations are generated by $H_\psi$ acting as Hamiltonian in an appropriate Poisson structure. Note that abstract versions of such equations on $N$ copies of a Lie group were written down in [10], but we restrict to (7) as a first step towards understanding the general case.

*Remark.* The flow on $(g_1, g_2^{-1})$ defined by (7) in the case where $\psi(x) = -\frac{1}{2}\mathrm{tr}(\log x)^2$, $x \in \mathbf{G}$, is a continuous time interpolation of the $QZ$ algorithm [2, 8] to compute generalized eigenvalues of the pair $(g_1, g_2^{-1})$, $(g_1 - \lambda g_2^{-1})u = 0$, as in [4]. On the other hand, we can also interpret the Lax system in (7) as an algorithm to compute the eigenvalues of the monodromy matrix $T(g) = g_2 g_1$. For the same reason, Lax systems on $\mathbf{G}^N = \mathbf{G} \times \cdots \times \mathbf{G}$ ($N$ copies) are of relevance for the computation of the eigenvalues of the product of $N$ matrices.

## 2

We now introduce the Poisson structure and the integrals for (7). In contrast to many eigenvalue algorithms, where the underlying Hamiltonian structures are linear [4], the relevant Poisson structure here is given by the quadratic Poisson structure (see [7])

$$\begin{aligned} &\{\varphi_1, \varphi_2\}(g_1, g_2) \\ &\equiv \frac{1}{2}\sum_{j=1}^{2}[\langle A(D_j'\varphi_1), D_j'\varphi_2\rangle - \langle A(D_j\varphi_1), D_j\varphi_2\rangle \\ &\quad + \langle S(D_j\varphi_1), D_{j-1}'\varphi_2\rangle - \langle S(D_{j-1}'\varphi_1), D_j\varphi_2\rangle], \end{aligned} \tag{8}$$

where $A = \frac{1}{2}(R - R^*)$, $S = \frac{1}{2}(R + R^*)$ and the indices are taken mod 2. This construction is an extension of the result in [10] and it is easy to show that the eigenvalues of the monodromy matrix Poisson commute in $\{\ ,\ \}$. Moreover, $\{\cdot, H_\psi\}$ generates (7). In what follows, we shall describe the generic symplectic leaves of this Poisson structure and the additional integrals necessary to show that (7) is integrable on such leaves.

The integrals fall into two categories according to their invariance properties (see also §3). The first set comes from the coefficients of the polynomial

$$(9)\qquad \begin{aligned} J(g_1, g_2; h, z) &\equiv \det(g_2 g_1 - h g_2 g_2^T - z)^* \\ &= \sum_{r=0}^{n}\sum_{k=0}^{r} J_{rk}(g_1, g_2) h^k z^{n-r}, \qquad (g_1, g_2) \in \mathbf{G}^2 \end{aligned}$$

so that there is an associated algebraic (spectral) curve

$$J(g_1, g_2; h, z) = 0.$$

For an $n \times n$ matrix $M$, let $(M)_k$ denote the $(n-k) \times (n-k)$ matrix obtained from $M$ by deleting the first $k$ columns and last $k$ rows. For $(g_1, g_2) \in \mathbf{G}^2$, we put

$$(10)\quad P_k(g_1, g_2; \lambda) \equiv \det(g_1 - \lambda g_2^{-1})_k = \sum_{r=0}^{n-k} E_{rk}(g_1, g_2)\lambda^{n-k-r}, \qquad k = 0, 1, \dots, n-1,$$

and note that the signs of $\det(g_2)_k$, $k = 0, 1, \dots, n-1$ are constant on the symplectic leaves of $\{\ ,\ \}$. To introduce the second set of invariants, we make the genericity hypothesis

$$\text{(G1)}\qquad \det(g_2)_k \neq 0, \qquad k = 1, \dots, n-1.$$

This allows us to define, for $g_2 \in \mathbf{G}$ satisfying (G1), the integrals

$$(11)\quad I_{rk}(g_1, g_2) \equiv E_{rk}(g_1, g_2)/E_{0k}(g_1, g_2), \qquad 0 \le k \le n-1, \quad 1 \le r \le n-k.$$

In order to obtain symplectic leaves of maximal dimension, we further impose

$$\text{(G2)}\qquad \sigma(g_1, g_2^T) = \{z \in \mathbf{C} \mid \det(g_1 - z g_2^T) = 0\}$$

is simple.

**Theorem.** (*a*) *Let* $(g_1, g_2) \in \mathbf{G}^2$ *satisfy* (G1) - (G2). *Then the symplectic leaf* $\mathscr{L}_{(g_1, g_2)}$ *of the Poisson structure given in* (8) *passing through the point* $(g_1, g_2)$ *is of dimension* $2n(n-1)$, *being the level set of the Casimir functions*** $J_{nk}$, $0 \le k \le n$ *and* $I_{n-k,k}$, $1 \le k \le n$.

(b) *The functions* $I_{rk}$, $1 \le k \le n-1$, $1 \le r \le n-k-1$ *and* $J_{r'k'}$, $1 \le r' \le n-1$, $0 \le k' \le r'$ *provide* $n(n-1)$

* For a matrix $M$, $M^T$ is the transpose of $M$.

** A Casimir function on a Poisson manifold is a function $C$ such that $\{C, F\} = 0$ for all functions $F$.

*independent Poisson commuting integrals for* (7) *on an open dense set in* $\mathscr{L}_{(g_1, g_2)}$.

(c) *Let* $Q_i(t) \in O^+(n, \mathbf{R})$, $L_i(t) \in L^+(n, \mathbf{R})$, $i = 1, 2$, *be solutions of the factorization problem*

$$e^{-tD_1 I_{rk}(g_1^0 c, g_2^0)} = Q_1(t)L_1(t), \qquad e^{-tD_1' I_{rk}(g_1^0, g_2^0)} = Q_2(t)L_2(t).$$

*Then the solution to the* $I_{rk}$*-flow is given by*

$$g_1(t) = Q_1^T(t)g_1^0 Q_2(t), \qquad g_2(t) = Q_2^T(t)g_2^0 Q_1(t).$$

(d) *The* $J_{rk}$*-flow,* $k > 0$, *induces an isospectral flow on* $X_h(g_1, g_2) = g_2 g_1 - h g_2 g_2^T$, *given by the equation*

$$\dot{X}_h = [X_h, B^+(h)] = [X_h, B^-(h)],$$

*where* $B^+(h)$, $B^-(h)$ *are matrix polynomials in* $h$ *and* $h^{-1}$, *respectively, with* $B^+(0) \in \mathbf{l}$, $B^-(\infty) \in \mathbf{k}$, *and* $B^+(h) - B^-(h) = h^{-k} X_h \nabla^T E_r(X_h)$ ($E_r$ *is the r-th elementary symmetric function). For an open dense set of initial data, this can be solved via a factorization problem in the loop group* $LGL(n, \mathbf{C})$.

*Remark.* The Symes type formula (cf. [11]) in part (c) of the theorem in the case where $k = 0$ is due to Semenov-Tian-Shansky [10] and Chu [2].

## 3

We give some ideas of the proof. Introduce the parabolic subgroups

$$\mathbf{G}_k = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \middle| A \in GL(n-k, \mathbf{R}),\ C \in GL(k, \mathbf{R}) \right\}, \qquad k = 1, \dots, n-1.$$

For (a) and (b), the basic facts to note are the invariance properties of the $I_{rk}$'s and $J_{rk}$'s:

(i) $I_{rk}(h_1 g_1 h_2^{-1}, h_2 g_2 h_1^{-1}) = I_{rk}(g_1, g_2)$, $\quad h_1 \in \mathbf{G}_k$, $\quad h_2 \in \mathbf{G}_{n-k}$,

(ii) $J_{rk}(h_1 g_1 h_2^{-1}, h_2 g_2 h_1^{-1}) = J_{rk}(g_1, g_2)$, $\quad h_1, h_2 \in O(n, \mathbf{R})$,

together with the observation that

(iii) $\quad D_1 I_{rk} \in \mathbf{g}_k = \mathrm{Lie}(\mathbf{G}_k)$, $\qquad D_1' I_{rk} \in \mathbf{g}_{n-k} = \mathrm{Lie}(\mathbf{G}_{n-k})$.

In addition, the explicit form of the $J_{rk}$'s is crucial. The open dense set in $\mathscr{L}_{(g_1, g_2)}$ where the integrals are independent consists of elements satisfying additional properties (besides (G1) and (G2)) which we shall not try to explain here. For (c), verification in the case $k > 0$ requires finer information about $Q_1(t)$ and $Q_2(t)$, namely,

$$Q_1^T(t) \in \mathbf{G}_k \cap O^+(n, \mathbf{R}), \qquad Q_2^T(t) \in \mathbf{G}_{n-k} \cap O^+(n, \mathbf{R}).$$

Finally, the factorization problem in the loop group $LGL(n, \mathbf{C})$ can be reduced to a scalar factorization problem, as in [3, 9]. We shall report the details of this work elsewhere.


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