BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Huygens' principle and hyperbolic equations, by Paul Günther. Perspectives in Mathematics, vol. 5, Academic Press, San Diego, 1988, viii + 847 pp., $69.00. ISBN 0-12-307330-8

Each of the three prominent families of linear partial differential equations: elliptic, parabolic, and hyperbolic, has its own world of mathematical properties and corresponding physical interpretations. Perhaps the hyperbolic equations remain the most mysterious. They often describe the propagation of waves in space and time, and the physical ideas surrounding this situation are extremely powerful guides to questions such as existence and uniqueness as well as qualitative details for the solutions.

In the century where science was bursting out from the long hibernation of the Middle Age, Christian Huygens (1629–1695) put forward the wave theory of light, later to be further developed with Newton. Huygens found that such waves propagated along wave fronts, and that each point on the front acted as a point source for a new wave. Thus he could calculate actual progressing waves by superposition. In a modern formulation, this amounts to the linear hyperbolic character, in particular the finite propagation speed, of solutions to the Maxwell equations.

To explain what is meant today by Huygens' principle for a hyperbolic equation (sometimes called the strong Huygens' principle), we could simply define it to mean that sharp signals propagate as sharp signals. for example, in flat Minkowski space $\mathbf{R} \times \mathbf{R}^3$ the scalar wave equation for $u = u(t, x_1, x_2, x_3)$,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0 \tag{1}$$

admits radial solutions $u = r^{-1} f(r - t)$, where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, and f say is smooth of compact support in $(0, \infty)$. Clearly these radial solutions propagate (in fact, they radiate away from

$r = 0$ as time t increases) in such a way that

$$u = \frac{\partial u}{\partial t} = 0 \quad \text{at} \quad t = 0 \qquad \text{for } r > R$$

(2) implies

$$u = 0 \quad \text{at} \quad t > r + R.$$

(Actually they do even better: they vanish at $t \geq r$.) If (2) holds for all solutions u of a hyperbolic equation (or system) in $\mathbf{R} \times \mathbf{R}^3$, then we say that the differential equation/operator in question satisfies Huygens' principle. This is indeed the case for (1), a fact which requires some not quite elementary integration and distribution theory (one may for example superimpose radial solutions, radiating out from different points). We shall be a little more specific about this example next.

The equation (1), generalized to m-dimensional Minkowski space $\mathbf{R} \times \mathbf{R}^{m-1}$, has a fundamental solution

$$F_+ = \frac{\pi^{(2-m)/2}}{2\Gamma\left(\frac{2-m}{2} + 1\right)} \cdot \gamma^{(2-m)/2} \cdot \chi_+ \tag{3}$$

satisfying $\Box F_+ = \delta$, where $\Box$ is the scalar wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_{m-1}^2} \tag{4}$$

and δ is the Dirac distribution at the origin. γ denotes the Lorentz metric square $t^2 - x_1^2 - \cdots - x_{m-1}^2$, and χ_+ the characteristic function of the forward light cone: $\gamma > 0$ and $t > 0$. F_+ may be defined by analytic continuation (in the dimension m, no less,using the properties of $\Gamma(k) = (k-1)!$) as the value of a meromorphic family of distributions. F_+ is called the advanced fundamental solution, and its time-reverse F_- is called the retarded fundamental solution. The main fact is that:

(5) For m even the support of F_+ is contained in the boundary of the forward light cone, i.e. in $\gamma = 0$ and $t \geq 0$.

When $m = 4$, this is simply

$$F_+ = \frac{1}{4\pi} \frac{\delta(t-r)}{r}.$$

By using Green's theorem over a region bounded by $t = 0$ and the boundary of the backward light cone from a point $(t_1, 0, \dots, 0)$ with $t_1 > 0$, it is seen that (5) implies Huygens' principle. On the other hand, in odd-dimensional Minkowski space, F_+ is supported in the solid cone, and what happens for solutions at $(t_1, 0, \dots, 0)$ depends on the Cauchy data u and $\partial u/\partial t$ for $t = 0$ at all the points with $r \leq t_1$.

The above summarizes the situation for the standard example, where our four-dimensional Minkowski space is a good place to send and receive wave signals governed by (1); on the other hand, "flatland", where $m = 3$, would be a noisy and forever echoing mess of old and new signals. We also see how to generalize the notion of Huygens' principle to the setting of general hyperbolic operators on curved manifolds: With a time coordinate present either via sharp propagation as in (2), or else via (5) as a special property of fundamental solutions having support in a singular subvariety connected with an underlying causal structure on the manifold [10].

Examples to keep in mind are the curved analogues of (4): Laplace–Beltrami operators on Lorentz manifolds with possibly lower-order terms. The first investigations on these issues are due to J. Hadamard (1865–1963) who created the main technical tools, still in effective use today.

It may seem a very special task to investigate Huygens' principle —is it worth studying and devoting a lot of effort to what is surely a rare class of partial differential operators? Even though we are far from having a classification of Huygens' operators, and what we do have amounts to little more than a list of examples, I still have no doubt it is an important field of inquiry. It is as if you are pulling on a small line connected to a network set up for bigger game: (i) wave fronts and propagation of singularities for hyperbolic equations, (ii) the Cauchy problem for hyperbolic equations, (iii) Riemannian geometry, in particular invariant theory and spectral geometry, (iv) conformal differential geometry, (v) Lie group theory connected with the metaplectic representation [7], and (vi) wave propagation in symmetric spaces [6].

The book of P. Günther is the result of a long life with Huygens' principle—indeed, it almost has the character of very careful notes taken over the course of a thirty-five-year career, collecting and clarifying insights. We are not faced here with an architecturally grand mathematical theory, where everything fits in place;

but rather with a subject that gets its life from details and examples, from tedious calculations with tensors and indices (and they work!), and from the eventual tie-up with basic facts in Minkowski space.

In this spirit I would like to present to the reader a few technicalities representative of dealing with Huygens' principle, and after that comment on Günther's book in a little more detail.

Let M be an m-dimensional Lorentz manifold, i.e. admitting a metric tensor g of signature $(1, -1, \dots, -1)$. Consider a second-order differential operator P acting on the sections u of a vector bundle E over M, such that the leading term is given by the metric:

$$Pu^I = \sum_{ij} g^{ij} \frac{\partial^2 u^I}{\partial x_i \partial x_j} + \text{lower order}, \tag{6}$$

where (g^{ij}) denotes the inverse of the metric tensor (g_{ij}) with respect to local coordinates $(x_1, \dots, x_m)$; the index I indicates that u is vector-valued. A convenient characterization of this class of operators (6), called normal hyperbolic operators, is

$$P = B + C, \tag{7}$$

where B is the Bochner Laplacian corresponding to a connection D on E: $B = \sum_{i,j} g^{ij} D_i D_j$, and C is a zero-order operator on the sections of E. With R the scalar curvature, the Cotton tensor is defined to be

$$c = C + \frac{m-2}{4(m-1)} \cdot R. \tag{8}$$

To calculate a local formula for an approximate forward fundamental solution F_+, one uses the Hadamard approach: For y in a nice (geodesically normal) neighborhood of x in M, consider the quadratic geodesic distance function $\Gamma(x, y)$ analogous to γ above in Minkowski space. Then there is an expansion for F_+ in terms of powers of $\Gamma = \Gamma(x, y)$, where the coefficients $\mathscr{U}_{(k)}(x, y)$ are Hom(E)-valued kernels. For m even it is (modulo some more technicalities involving cut-off functions and measure densities)

$$F_+ = \sum_{k=0}^{\infty} \mathscr{U}_{(k)}(x, y) \Gamma^{k+(2-m)/2}. \tag{9}$$

This splits into $F_+ = V_{\text{singular}} + V_{\text{regular}}$, the singular part involving negative powers of Γ, and the regular part being the remaining sum from $k = (m-2)/2$ to infinity. (9) is to be understood as an asymptotic expansion in Γ, and the Hadamard coefficients $\mathscr{U}_{(k)}$ are determined recursively by certain first-order ordinary differential equations along geodesics (the so-called transport equations). V_{singular} is so normalized as to have support on the boundary of the future of x, i.e. on the subvariety generated by forward null geodesics (light rays). Of course, all these assertions are localized at x. V_{regular} is sometimes referred to as the tail of the fundamental solution; it is a distribution supported inside the future of x. Note again that we are working locally: Huygens' principle could hold for P in certain regions of M and fail in others.

Some technical care must be devoted to the convergence questions in (9); it is only an equality in the sense of asymptotic expansions; but a suitable technical fix introduced by E. Borel allows one to perform some of the needed manipulations as if we were dealing with a convergent series. Sweeping these things under the carpet, we proceed by observing that Huygens' principle will hold (in the forward sense) if and only if the regular part of F_+ is zero, that is, provided

$$\mathscr{U}_{(m-2)/2}(x, y) = \text{ identically.} \tag{10}$$

Note that by the transport equations all the following Hadamard coefficients will also vanish, and only the singular part of F_+ survives. (10) is just another expression of the "vanishing of the log-term" introduced by Hadamard.

Equation (10) is the main focus of the energies of Günther and his students. In particular they have developed a theory of so-called moments, which are local invariants corresponding to the Taylor expansion around $y = x$ of the tail end Hadamard coefficients. These moments are built tensorically from the Riemannian curvature, the bundle curvature, the Cotton tensor c from (8), and their covariant derivatives. Huygens' principle implies the vanishing of all these moments (and conversely in the analytic case), which by construction all transform in a simple way under conformal gauge transformations: replacing the metric by $\tilde{g} = e^{2\varphi}g$, φ a real function on M, and the operator P by $\tilde{P}u = e^{-(m+2)\varphi/2}P(e^{(m-2)\varphi/2}u)$. One of the good things in all this is that the Cotton tensor then transforms nicely: $\tilde{c} = e^{-2\varphi}c$.

In his book, Günther concentrates on the class (6) and (7) and calculates enough moments to (a) give examples of Huygens' principle, (b) give necessary conditions on an operator for Huygens' principle to hold, (c) give necessary and sufficient conditions on the metric in order for Huygens' principle to hold for such natural equations as the conformal wave equation:

$$\sum_{i,j} g^{ij}\nabla_i\nabla_j u - \frac{m-2}{4(m-1)} \cdot Ru = 0$$

(also known as the Yamabe equation) and Maxwell's equations on differential forms ω:

$$d\omega = 0, \qquad \delta\omega = 0$$

with δ the adjoint of the exterior derivative d. (Here it is noted that Maxwell's equations do not satisfy Huygens' principle in the famous Schwartzschild metric for "black holes"—a matter for cosmologists?)

The book presents many interesting examples of Huygens' principle, for instance the equations discovered by Helgason [6]

$$\frac{\partial^2 u}{\partial t^2} - \left(\Delta + \frac{R}{6}\right) u = 0, \tag{11}$$

where Δ is the Laplace–Beltrami operator on either a simple compact Lie group or on G/K for G complex and simple with K a maximal compact subgroup. Günther's approach to (11) is to study the spacial elliptic part $\Delta + R/6$ of (11) and show that it satisfies a special vanishing ("property E") of its Hadamard coefficients. In this connection is offered a most appealing conjecture for equations of the general form

$$\frac{\partial^2 u}{\partial t^2} + Pu = 0 \quad \text{on } \mathbf{R} \times M_1. \tag{12}$$

Here P is a positive elliptic operator on an odd-dimensional compact Riemannian manifold M_1. Could Huygens' principle for (12) correspond to equality in the Duistermaat–Guillemin formula [3], i.e. (say P is leading metric)

$$tr e^{itP^{1/2}} = \sum_l \delta_l,$$

where δ_l are the δ-distributions supported at the lengths of closed geodesics?

The book by Günther develops in a concise way the local invariant theory behind any serious investigation of Huygens' principle. I enjoyed seeing the criteria unfold into nice examples especially for the particular class of metrics of plane-wave type. Here the theory is at its best in terms of giving the connection between Huygens' principle and the geometry.

This having been said, I also see a few shortcomings in the book. There is very little mention of other approaches to the construction of fundamental solutions and to the invariant theory. For the latter, one finds strangely no reference to the work of P. Gilkey [5], or Fefferman–Graham [4], for example. These and other recent developments would have fitted in nicely, as would at least a mention of the Seeley calculus [9]. Indeed, one would expect modern tools such as pseudo-differential operators, Fourier-integral operators and the study of wave fronts to cast even more light on Huygens' principle. Also missing are operators not of normal hyperbolic type (apart from Maxwell and Dirac). Here T. Branson [1] has found several classes of Huygens' operators based on $ad\delta + b\delta d$ with $a \neq b$, acting on differential forms. Furthermore, in a recent preprint [2], Branson and 'Olafsson suggest a correspondence between Huygens' principle and the physical principle of equipartition of energy for wave equations. Finally, the book misses some recent developments, such as the beautiful characterization of Huygens' principle by decay along geodesics by K. Nishiwada [8].

Apart from very diligent graduate students (who can get on without a good index, exercises and general guidance), Günther's book addresses both geometers and analysts working in partial differential equations. Here they will find an attractive common scene where the insights of Christian Huygens are still worth building on: but now the cast includes invariant theory on manifolds, fundamental solutions, and, mysteriously, conformal geometry.

References

1. T. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), 199–291.
2. T. Branson and G. 'Olafsson, *Equipartition of energy for waves in symmetric space*, preprint, 1989.
3. J. J. Duistermaat and V. Guillemin, Invent. Math. **29** (1975), 39–80.
4. C. Fefferman and C. R. Graham, *Conformal invariants*, in Elie Cartan et les mathématiques d'aujourd'hui, Astérisque (1985), 95–116.
5. P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish, Wilmington, Delaware, 1984.
6. S. Helgason, *Wave equations on homogeneous spaces*, Lecture Notes in Math., vol. 1077, Springer-Verlag, Berlin and New York, 1984, pp. 254–288.

7. R. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Amer. Math. Soc. **3** (1980), 254–845.
8. K. Nishiwada, *Huygens' principle for a wave equation and the asymptotic behavior of solutions along geodesics*, in Hyperbolic equations and related topics (S. Mizohata, ed.), Academic Press, 1986.
9. R. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure Appl. Math. **10** (1976), 288–307.
10. I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy*, Academic Press, New York, 1976.

BENT ØRSTED
ODENSE UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Existentially closed groups, by Graham Higman and Elizabeth Scott. London Mathematical Society Monographs, New Series, vol. 3., Clarenden Press, Oxford, 1988, xiv + 156 pp., $49.95. ISBN 0-19-853543-0

SUMMARY

This volume grew out of lectures given by Higman at Oxford in 1983 and 1984 as recorded and amended by Scott. It is not a comprehensive work on e.c. groups but rather contains an ample selection of topics written at an easily accessible graduate level. Both algebraic and model-theoretic aspects of e.c. groups are highlighted. Thus, Chapter 2 gives two very diferent group-theoretic proofs that the normalizer of a finite characteristically simple subgroup of an e.c. group G is a maximal subgroup of G, as well as related results, and has considrable technical interest. [For extensions of one of these methods, see the reviewer's "A.c. groups: Extensions, maximal subgroups, and automorphisms," Trans. Amer. Math Soc. 290, (1985), 457–481.] This book contains all the results of Hickin and Macintyre's "A.c. groups: Embeddings and centralizers" (in *Word Problems* II, North-Holland, 1980) with the exception of the spectrum problem in power ω_1. After some preliminaries, Chapters 5 and 6 develop some algebraic applications of the Higman embedding theorem and its generalized version (which is deduced in the text). In particular the embedding of