Sierpiński, W., *Leçons sur les nombres transfinis*, Monographies Emile Borel, Paris, 1928.

Szpilrajn, E., *Sur l'extension de l'ordre partiel*, Fund. Math. **16** (1930), 386–389.

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§1. INTRODUCTION

During the past decades the *mathematical* theory of nonlinear three-dimensional elasticity has undergone a considerable renewed interest, reflected for instance by the books of Marsden and Hughes [17], Ciarlet [8], and the book reviewed here.

The existence results available at the present time fall in two categories:

In one approach (described in §§2 and 5) the problem is posed as a *system of three quasilinear partial differential equations of the second order*, together with specific boundary conditions (cf. (13)), and one tries to obtain *"local" existence results based on the implicit function theorem*; this approach, which was initiated by Stoppelli [18], is the central theme of the book under review.

In another approach (described in §§3 and 4), the problem is posed as a *minimization problem for the associated energy* (cf. (20)), and one tries to adapt the paraphernalia of the calculus of variations (infimizing sequences, weak convergence, weak lower semi-continuity, etc.) to this problem, which is "highly nonconvex"; this approach is the basis of a famous existence result of Ball [3].

All these results apply to "static" equilibria, i.e. to problems that are *time-independent*. While substantial progress has thus been made in the study of statics, the mathematical analysis of time-dependent three-dimensional elasticity still meets with inextricable difficulties. The proofs of the available existence results "for large times," even though they apply to only one space variable, are already exceedingly delicate.

§2. THREE-DIMENSIONAL ELASTICITY

Detailed expositions of the mathematical modeling of three-dimensional elasticity are found in Truesdell and Noll [19], Wang and Truesdell [23], Gurtin [14], Marsden and Hughes [17, Chapters 1–5], Ciarlet [8, Chapters 1–5].

The central problem in nonlinear, three-dimensional, static elasticity consists in finding the equilibrium position of an elastic body when it is subjected to applied forces. This body occupies a *reference configuration* $\overline{\Omega}$ in the absence of forces, where Ω is a domain in $\mathbf{R}^3$, i.e. a bounded, connected, open subset of $\mathbf{R}^3$ with a Lipschitz-continuous boundary Γ; in particular then, a unit normal vector $\mathbf{n} = (n_i)$ exists almost everywhere along Γ.

When subjected to *applied forces*, the body occupies a *deformed configuration* $\boldsymbol{\varphi}(\overline{\Omega})$, where the mapping $\boldsymbol{\varphi}: \overline{\Omega} \to \mathbf{R}^3$, which is called a *deformation*, must be orientation-preserving in the set $\overline{\Omega}$ and injective on the set Ω, in order to be physically acceptable (the reason a deformation need not be injective on $\overline{\Omega}$ is that self-contact must be allowed).

Let $\mathbf{M}^3$ denote the set of all real matrices of order 3 and let

$$\mathbf{M}^3_+ = \{\mathbf{F} \in \mathbf{M}^3 ; \det \mathbf{F} > 0\}.$$

Then the orientation-preserving character of a deformation imposes that its deformation gradient $\nabla\boldsymbol{\varphi}(x)$, defined by

$$\nabla\boldsymbol{\varphi} = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 & \partial_3\varphi_2 \\ \partial_1\varphi_3 & \partial_2\varphi_3 & \partial_3\varphi_3 \end{pmatrix}, \qquad \text{where } \partial_i = \frac{\partial}{\partial x_i},$$

be in the set $\mathbf{M}^3_+$ for all $x \in \overline{\Omega}$.

A body occupying a deformed configuration $\boldsymbol{\varphi}(\overline{\Omega})$, and subjected to applied body forces in its interior and to applied surface forces on a portion $\boldsymbol{\varphi}(\Gamma_1)$ of its boundary, where Γ_1 is a subset of Γ, is in *static equilibrium* if the fundamental *stress principle of Euler and Cauchy* is satisfied. This axiom implies the celebrated *Cauchy theorem*, according to which:

(i) There exists a tensor field $\mathbf{T}: \overline{\Omega} \to \mathbf{M}^3$ that satisfies the *equilibrium equations over the reference configuration*:

$$(1) \qquad \begin{cases} -\mathbf{div}\,\mathbf{T}(x) = \hat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), & x \in \Omega, \\ \mathbf{T}(x)\mathbf{n}(x) = \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), & x \in \Gamma_1. \end{cases}$$

or, componentwise,

$$(2)\quad \begin{cases} -\sum_{j=1}^{3} \partial_j T_{ij}(x) = \widehat{f}_i(x, \boldsymbol{\varphi}(x)), & x \in \Omega,\ 1 \le i \le 3, \\ \sum_{j=1}^{3} T_{ij}(x) n_j(x) = \widehat{g}_i(x, \boldsymbol{\nabla\varphi}(x)), & x \in \Gamma_1,\ 1 \le i \le 3. \end{cases}$$

(ii) The tensor $\mathbf{T}(x) = (T_{ij}(x))$, which is called the *first Piola-Kirchhoff stress tensor* at the point $x \in \Omega$, is the *Piola transform*

$$(3)\qquad \mathbf{T}(x) = (\det \boldsymbol{\nabla\varphi}(x))\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))\boldsymbol{\nabla\varphi}(x)^{-T}$$

of the *Cauchy stress tensor* $\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))$ at the point $\boldsymbol{\varphi}(x)$; since the Cauchy stress tensor is symmetric, one thus has, by (3),

$$(4)\qquad \mathbf{T}(x)\boldsymbol{\nabla\varphi}(x)^T \in \mathbf{S}^3 \qquad \text{for all } x \in \overline{\Omega},$$

where $\mathbf{S}^3 = \{\mathbf{A} \in \mathbf{M}^3;\ \mathbf{A} = \mathbf{A}^T\}$.

The mappings $\widehat{\mathbf{f}}\colon \Omega \times \mathbf{R}^3 \to \mathbf{R}^3$ and $\widehat{\mathbf{g}}\colon \Omega \times \mathbf{M}^3_+ \to \mathbf{R}^3$ respectively measure the *density of the applied body force* per unit volume, and the *density of the applied surface force* per unit area, in the reference configuration. For instance, a *pressure load* corresponds to a density $\widehat{\mathbf{g}}$ of the form

$$(5)\quad \widehat{\mathbf{g}}(x, \boldsymbol{\nabla\varphi}(x) = -\pi(\det \boldsymbol{\nabla\varphi}(x))\boldsymbol{\nabla\varphi}(x)^{-T}\mathbf{n}(x), \qquad x \in \Gamma_1,$$

where π is a real constant, called the *pressure*; the *gravity field* corresponds to a density $\widehat{\mathbf{f}}$ of the form

$$(6)\qquad \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) = -g\rho(x)\mathbf{e}_3, \qquad x \in \Omega,$$

where g is the gravitational constant, $\rho(x)$ is the mass density, and $\mathbf{e}_3$ is the "third" basis vector, assumed to be "vertical" and "upward oriented." The density in (6) is that of a "dead load": an applied body or surface force is a *dead load* if its density is independent of the particular deformation $\boldsymbol{\varphi}$ considered. Note in passing that actual applied forces can seldom be modeled as dead loads (except the gravity field, or a pressure load with $\pi = 0$!).

The equations of equilibrium (1) must clearly be complemented by equations that specify the nature of the constituting material that is considered; the undetermination of equations (1) is also clear from a mathematical standpoint, since there are *three* equations in (2) and *nine* unknowns, the three components of the deformation and the six independent components of the tensor $\mathbf{T}$ (there are three equations in relation (4)).

A material is *elastic* if, at each point $\boldsymbol{\varphi}(x)$ of the deformed configuration, the Cauchy stress tensor $\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))$ is solely a function of x and of the deformation gradient $\boldsymbol{\nabla\varphi}(x)$. Equivalently, by

(3), a material is elastic if, at each point $x \in \overline{\Omega}$, the first Piola–Kirchhoff stress tensor is expressed in terms of x and $\nabla\varphi(x)$ through a *constitutive equation* of the form

$$\text{(7)} \qquad \begin{aligned} \mathbf{T}(x) &= \widehat{\mathbf{T}}(x, \nabla\varphi(x)), \quad \text{or equivalently} \\ T_{ij}(x) &= \widehat{T}_{ij}(x, \nabla\varphi(x)), \qquad 1 \le i,\ j \le 3, \end{aligned}$$

where the *response function*

$$\text{(8)} \qquad \widehat{\mathbf{T}} = (\widehat{T}_{ij}) : (x, \mathbf{F}) \in \overline{\Omega} \times \mathbf{M}^3_+ \to \widehat{\mathbf{T}}(x, \mathbf{F}) \in \mathbf{M}^3$$

characterizes the elastic material. Note that, by (4), the response function must also satisfy

$$\text{(9)} \qquad \widehat{\mathbf{T}}(x, \mathbf{F})\mathbf{F}^T \in \mathbf{S}^3 \qquad \text{for all } (x, \mathbf{F}) \in \overline{\Omega} \times \mathbf{M}^3_+.$$

One must then take into account the *axiom of material frame-indifference*, a general principle in physics that, loosely speaking, asserts that any observable quantity with an "intrinsic" character (here, the Cauchy stress vector) must be independent of the particular basis in which it is computed. As expected, the effect of this axiom, also known as the *axiom of invariance under a change of observer*, or the *axiom of objectivity*, is to reduce the class of mappings of the form (8), and which satisfy (9), that may be used in a constitutive equation (7) of an elastic material.

More specifically, let $\mathbf{O}^3_+$ denote the set of all rotations in $\mathbf{R}^3$, i.e., orthogonal matrices $\mathbf{Q}$ of order 3 with $\det \mathbf{Q} = +1$. Then an elastic material is frame-indifferent if and only if, at each $x \in \overline{\Omega}$,

$$\text{(10)} \qquad \widehat{\mathbf{T}}(x, \mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(x, \mathbf{F}) \qquad \text{for all } \mathbf{Q} \in \mathbf{O}^3_+,\ \mathbf{F} \in \mathbf{M}^3_+.$$

As shown by Fosdick and Serrin [13], a noteworthy consequence of relation (10) is that *the response function* $\widehat{\mathbf{T}}$ *cannot be linear with respect to its argument* $\mathbf{F} \in \mathbf{M}^3_+$ if the reference configuration is a *natural state*, i.e. if the stress tensor vanishes when $\varphi = \mathrm{id}$ (this is equivalent to saying that $\widehat{\mathbf{T}}(x, \mathbf{I}) = \mathbf{0}$ for all $x \in \overline{\Omega}$). Note in passing that this observation definitely rules out *linear* partial differential equations (as in (13) below) as a possible model of elasticity!

Let us assume that the unknown deformation φ satisfies a *boundary condition of place*, of the form

$$\text{(11)} \qquad \varphi(x) = \varphi_0(x), \qquad x \in \Gamma_0,$$

where $\varphi_0 : \Gamma_0 \to \mathbf{R}^3$ is a given mapping, on the remaining portion $\Gamma_0 = \Gamma - \Gamma_1$ of the boundary of Ω. We recall (cf. (1)) that on

Γ_1, $\boldsymbol{\varphi}$ satisfies a *boundary condition of traction*, of the form

$$(12)\qquad \mathbf{T}(x)\mathbf{n}(x) = \widehat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), \qquad x \in \Gamma_1.$$

Note that the boundary conditions of place and traction far from exhaust all the situations occurring in practice, where in particular *unilateral* boundary conditions are quite common.

Assembling the various notions found so far, we are thus seeking a deformation $\boldsymbol{\varphi}: \overline{\Omega} \to \mathbf{R}^3$ that solves the following *boundary value problem of three-dimensional elasticity* (we recall that a deformation must be orientation-preserving in $\overline{\Omega}$ and injective on Ω):

$$(13)\qquad \begin{cases} -\mathbf{div}\,\widehat{\mathbf{T}}(x, \nabla\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), & x \in \Omega, \\ \boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x), & x \in \Gamma_0, \\ \widehat{\mathbf{T}}(x, \nabla\boldsymbol{\varphi}(x))\mathbf{n}(x) = \widehat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), & x \in \Gamma_1. \end{cases}$$

This problem is called a *pure displacement problem* if $\Gamma_1 = \emptyset$, a *displacement-traction problem* if area $\Gamma_0 > 0$ and area $\Gamma_1 > 0$, and a *pure traction problem* if $\Gamma_0 = \emptyset$.

Assuming appropriate differentiability, we can write the equations in Ω found in (13) as

$$(14)\qquad \begin{aligned} &-\sum_{j,k,l=1}^{3} \frac{\partial \widehat{T}_{ij}}{\partial F_{kl}}(x, \nabla\boldsymbol{\varphi}(x)) \frac{\partial^2 \varphi_k}{\partial x_j \partial x_l}(x) \\ &-\sum_{j=1}^{3} \frac{\partial \widehat{T}_{ij}}{\partial x_j}(x, \nabla\boldsymbol{\varphi}(x)) = \widehat{f}_i(x, \boldsymbol{\varphi}(x)), \qquad 1 \le i \le 3. \end{aligned}$$

In the terminology of partial differential equations, such second-order equations, whose higher-order terms are nonlinear functions of $\nabla\boldsymbol{\varphi}(x)$, are labeled *quasilinear*, by contrast with *semilinear* equations, where the terms containing the partial derivatives of the highest order are linear.

Quasilinear partial differential equations are considerably harder to analyze than semilinear ones; this is one reason why so many difficulties are encountered in the mathematical analysis of three-dimensional elasticity.

§3. Hyperelastic materials

An elastic material is *hyperelastic* if there exists a *stored energy function* $\widehat{W}: \overline{\Omega} \times \mathbf{M}^3_+ \to \mathbf{R}$ such that

$$(15)\qquad \widehat{\mathbf{T}}(x, \mathbf{F}) = \frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) \qquad \text{for all } x \in \overline{\Omega},\ \mathbf{F} \in \mathbf{M}^3_+,$$

or, componentwise,

$$\widehat{T}_{ij}(x, \mathbf{F}) = \frac{\partial \widehat{W}}{\partial F_{ij}}(x, \mathbf{F}) \qquad \text{for all } x \in \overline{\Omega},\ \mathbf{F} \in \mathbf{M}^3_+,\ 1 \leq i,\ j \leq 3.$$

Applied body and surface forces are *conservative* if there exist *potentials* $\widehat{F}: \widehat{\Omega} \times \mathbf{R}^3 \to \mathbf{R}$ and $\widehat{G}: \Gamma_1 \times \mathbf{R}^3 \times \mathbf{M}^3_+ \to \mathbf{R}$ such that, for any smooth enough vector fields $\boldsymbol{\theta}: \overline{\Omega} \to \mathbf{R}^3$ that satisfy $\boldsymbol{\theta} = \mathbf{0}$ on Γ_0,

$$\int_\Omega \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)\, dx = F'(\boldsymbol{\varphi})\boldsymbol{\theta}, \qquad \text{with } F(\boldsymbol{\psi}) = \int_\Omega \widehat{F}(x, \boldsymbol{\psi}(x))\, dx, \tag{16}$$

$$\int_{\Gamma_1} \widehat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)\, dx = G'(\boldsymbol{\varphi})\boldsymbol{\theta}, \qquad \text{with } G(\boldsymbol{\psi}) = \int_{\Gamma_1} \widehat{G}(x, \boldsymbol{\psi}(x), \nabla\boldsymbol{\psi}(x))\, dx, \tag{17}$$

where $\cdot$ denotes the Euclidean inner product in $\mathbf{R}^3$, and $F'(\boldsymbol{\varphi})$ and $G'(\boldsymbol{\varphi})$ denote the Fréchet derivatives at $\boldsymbol{\varphi}$ of the functionals F and G. For instance, dead loads, or a pressure load, are conservative.

If the material is hyperelastic and if the applied forces are conservative, solving the boundary value problem (13) is formally equivalent to finding the *stationary points* of the *total energy* I, defined by

$$I(\boldsymbol{\psi}) = \int_\Omega \widehat{W}(x, \nabla\boldsymbol{\psi}(x))\, dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{\psi})\}, \tag{18}$$

when $\boldsymbol{\psi}$ varies in a *set of admissible deformations* of the form

$$\boldsymbol{\Phi} = \{\boldsymbol{\psi}: \overline{\Omega} \to \mathbf{R}^3;\ \boldsymbol{\psi} \text{ is injective on } \Omega,\ \det\nabla\boldsymbol{\psi} > 0 \text{ in } \overline{\Omega},\ \boldsymbol{\psi} = \boldsymbol{\varphi}_0 \text{ on } \Gamma_0\}. \tag{19}$$

In other words, the boundary value problem (13) forms the *Euler–Lagrange equations* associated with the total energy; in particular, any *minimizer* $\boldsymbol{\varphi}$ of the functional I over the set $\boldsymbol{\Phi}$, i.e., any $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ that satisfies

$$I(\boldsymbol{\varphi}) = \inf_{\boldsymbol{\psi}\in\boldsymbol{\Phi}} I(\boldsymbol{\psi}), \tag{20}$$

is a solution of this boundary value problem, provided it is smooth enough (note that a minimizer is a particular stationary point).

The axiom of material frame-indifference then implies that, at each point $x \in \overline{\Omega}$, the stored energy function $\widehat{W}(x, \cdot)$ is only a function of the *right Cauchy–Green strain tensor* $\{\nabla \boldsymbol{\varphi}(x)\}^T \nabla \boldsymbol{\varphi}(x)$. In other words, there exists a mapping $\widetilde{W}(x, \cdot)$ such that

$$\widehat{W}(x, \mathbf{F}) = \widetilde{W}(x, \mathbf{F}^T\mathbf{F}) \qquad \text{for all } \mathbf{F} \in \mathbf{M}^3_+. \tag{21}$$

The *behavior of the stored energy function for large strains*, which mathematically reflects the idea that "infinite stress must accompany extreme strains" [2], plays a crucial role in the existence theory: It takes the form of a *behavior* as $\det \mathbf{F} \to 0^+$:

$$\widehat{W}(x, \mathbf{F}) \to +\infty \quad \text{as } \det \mathbf{F} \to 0^+, \tag{22}$$

and of a *coerciveness inequality*: There exist constants $\alpha > 0$, $p > 0$, $q > 0$, $r > 0$, and β such that

$$\widehat{W}(x, \mathbf{F}) \geq \alpha\{\|\mathbf{F}\|^p + \|\mathbf{Cof\,F}\|^q + (\det \mathbf{F})^r\} + \beta \qquad \text{for all } \mathbf{F} \in \mathbf{M}^3_+, \tag{23}$$

where $\mathbf{Cof\,F} = (\det \mathbf{F})\mathbf{F}^{-T}$ is the cofactor matrix of the matrix $\mathbf{F}$. That the matrix $\mathbf{F}$, the matrix $\mathbf{Cof\,F}$, and the scalar $\det \mathbf{F}$, appear in the right-hand side of the coerciveness inequality (23) reflects the facts that the matrix $\nabla \boldsymbol{\varphi}$ (through the right Cauchy–Green strain tensor $\nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi}$), the matrix $\mathbf{Cof} \nabla \boldsymbol{\varphi}$, and the scalar $\det \nabla \boldsymbol{\varphi}$, respectively govern the changes of lengths, surfaces, and volumes, associated with a deformation $\boldsymbol{\varphi}$.

Note that *the stored energy function* $\widehat{W}: (x, \mathbf{F}) \in \overline{\Omega} \times \mathbf{M}^3_+ \to \mathbf{R}$ *cannot be convex with respect to the variable* $\mathbf{F} \in \mathbf{M}^3_+$: such a convexity would contradict both the behavior (22) as $\det \mathbf{F} \to 0^+$ [1] and the axiom of material frame-indifference [12].

The lack of convexity of the stored energy function, together with the lack of convexity of the set $\mathbf{\Phi}$ of admissible deformation (cf. (19)), stood for a long while as major difficulties in the mathematical analysis of three-dimensional hyperelasticity.

§4. Existence theory based on the minimization of the energy

The question of *existence* of solutions to the nonlinear boundary value problems of three-dimensional elasticity can be approached in two ways:

One approach consists in applying the *implicit function theorem* to the nonlinear boundary value problem (13). The existence results obtained in this fashion constitute the core of the book under review; they are discussed in the next section.

Another approach consists in seeking the *minimizers of the total energy* $I(\boldsymbol{\psi})$ of (18) when $\boldsymbol{\psi}$ varies in a set $\boldsymbol{\Phi}$ of admissible deformations of the form (19). This approach was given a considerable momentum by J. Ball, who showed in a landmark paper [3] how to extend the usual methods of the calculus of variations to this problem, in spite of its "lack of convexity."

More specifically, his method consists in considering an infimizing sequence $(\boldsymbol{\varphi}^k)$ of the total energy (18) over a set $\boldsymbol{\Phi}$ of the form

$$(24)\quad \begin{aligned}\boldsymbol{\Phi} = \{\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)\,;\ \mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\psi} \in \mathbf{L}^q(\Omega),\ \det\boldsymbol{\nabla}\boldsymbol{\psi} \in L^r(\Omega),\\ \boldsymbol{\psi} = \boldsymbol{\varphi}_0 \text{ on } \Gamma_0,\ \det\boldsymbol{\nabla}\boldsymbol{\psi} > 0 \text{ a.e. in } \Omega\},\end{aligned}$$

where the exponents p, q, r, which are precisely those appearing in the coerciveness inequality (23), are sufficiently large (more precisely: $p \geq 2$, $q \geq p/(p-1)$, $r > 1$). By (23), the sequence $(\boldsymbol{\varphi}^k, \mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\varphi}^k, \det\boldsymbol{\nabla}\boldsymbol{\varphi}^k)$ is thus bounded in a reflexive Banach space; whence there exists a subsequence $(\boldsymbol{\varphi}^l, \mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\varphi}^l, \det\boldsymbol{\nabla}\boldsymbol{\varphi}^l)$ that weakly converges to an element $(\boldsymbol{\varphi}, \mathbf{H}, \delta) \in \mathbf{W}^{1,p}(\Omega) \times \mathbf{L}^q(\Omega) \times L^r(\Omega)$. Then one shows that, remarkably, one has precisely $\mathbf{H} = \mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\varphi}$ and $\delta = \det\boldsymbol{\nabla}\boldsymbol{\varphi}$; this is a special case of the general phenomenon of compensated compactness introduced in 1978, and then extensively studied, by F. Murat and L. Tartar.

The behavior (22) of the stored energy function as $\det\mathbf{F} \to 0^+$ then implies that $\det\boldsymbol{\nabla}\boldsymbol{\varphi} > 0$ a.e. in Ω, and thus that the weak limit of the infimizing sequence belongs to the set $\boldsymbol{\Phi}$ of (24).

To obviate the lack of convexity of the stored energy function, J. Ball has introduced the fundamental notion of *polyconvexity*: A stored energy function $\widehat{W}: \overline{\Omega} \times \mathbf{M}^3_+ \to \mathbf{R}$ is *polyconvex* if, for each $x \in \overline{\Omega}$, there exists a convex function

$$\mathbf{W}(x, \cdot): \mathbf{M}^3 \times \mathbf{M}^3 \times]0, +\infty[\to \mathbf{R}$$

such that

$$(25)\quad \widehat{W}(x, \mathbf{F}) = \mathbf{W}(x, \mathbf{F}, \mathbf{Cof}\,\mathbf{F}, \det\mathbf{F}) \qquad \text{for all } \mathbf{F} \in \mathbf{M}^3_+.$$

Polyconvexity is clearly a much weaker requirement than convexity (for instance, the function $\mathbf{F} \in \mathbf{M}^3_+ \to \det\mathbf{F}$ is polyconvex, but not convex!). Contrary to convexity, such an assumption does not conflict with any physical requirement and indeed, it is satisfied by realistic models, such as *Ogden's*, or *Mooney–Rivlin, elastic materials.* Note that actual stored energy functions are naturally given as functions of $\mathbf{F}$, $\mathbf{Cof\,F}$, and $\det \mathbf{F}$, since, as already observed, $\nabla\boldsymbol{\varphi}$, $\mathbf{Cof}\,\nabla\boldsymbol{\varphi}$, and $\det\nabla\boldsymbol{\varphi}$, respectively "measure" the changes of lengths, surfaces, and volumes associated with a deformation $\boldsymbol{\varphi}$.

Another crucial contribution of J. Ball is that the assumption of polyconvexity implies the sequential weak lower semi-continuity of the total energy I, *viz.*,

$$I(\boldsymbol{\varphi}) \leq \liminf_{l\to\infty} I(\boldsymbol{\varphi}^l),$$

from which it follows that $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ is a minimizer of the total energy. J. Ball obtains in this fashion existence theorems in the space $\mathbf{W}^{1,p}(\Omega)$, $p \geq 2$, for pure displacement, pure traction (with an additional condition in the set $\boldsymbol{\Phi}$ in this case), and displacement-traction problems.

The existence results of J. Ball have been subsequently extended so as to take into account *unilateral boundary conditions of place* of the form "$\boldsymbol{\varphi}(\overline{\Omega}) \subset B$," where B is a closed subset of $\mathbf{R}^3$ [10], and the *injectivity condition* "$\boldsymbol{\varphi}$ is injective in Ω" [4, 11]. J. Ball's theory, as well as these extensions, are also exposed in detail in Ciarlet [8, Chapter 7].

A *major open problem* in J. Ball's approach consists in stating sufficient conditions that would imply additional *regularity of the minimizers.* Because such conditions are lacking, it is not known whether, in some specific cases, such a minimizer could be a solution, even in a weak sense, of the associated Euler–Lagrange equations, i.e., of the boundary value problem of three-dimensional elasticity (13).

§5. Existence theory based on the implicit function theorem

The idea of using the implicit function theorem goes back to Stoppelli [18] and van Buren [5]. The first complete existence results have been independently obtained by Valent [20], Marsden and Hughes [16, pp. 204 ff.], and Ciarlet and Destuynder [9]. T. Valent has in particular pursued in depth this approach in various directions, which are described at length in the book under review.

Given a deformation $\boldsymbol{\varphi}$ of the reference configuration $\overline{\Omega}$, let $\mathbf{u}$ denote the associated *displacement*, which is the vector field $\mathbf{u}\colon \overline{\Omega} \to \mathbf{R}^3$ defined by

$$\boldsymbol{\varphi} = \mathbf{id}_{\overline{\Omega}} + \mathbf{u},$$

where $\mathbf{id}_A$ denotes in general the identity mapping of a set A.

Consider the particular pure displacement problem (we have $\nabla\boldsymbol{\varphi} = \mathbf{I} + \nabla\mathbf{u}$):

$$(26) \qquad \begin{cases} -\mathbf{div}\,\widehat{\mathbf{T}}(x, \mathbf{I} + \nabla\mathbf{u}(x)) = \mathbf{f}(x), & x \in \Omega \\ \mathbf{u}(x) = \mathbf{0}, & x \in \Gamma, \end{cases}$$

which thus corresponds to the particular boundary condition of place $\boldsymbol{\varphi} = \mathbf{id}_\Gamma$, and assume that the reference configuration is a natural state, i.e., the stress tensor $\mathbf{T}$ vanishes if $\boldsymbol{\varphi} = \mathbf{id}$. Then clearly problem (26) possesses the particular solution $\mathbf{u} = \mathbf{0}$ corresponding to $\mathbf{f} = \mathbf{0}$ (for simplicity, we assume that the density $\mathbf{f}$ is that of a dead load).

Under mild smoothness assumptions on the response function $\widehat{\mathbf{T}}$, it can be shown that the *operator of nonlinear elasticity* $\mathbf{A}: \mathbf{u} \to \mathbf{A}(\mathbf{u})$, defined by

$$(27) \qquad \mathbf{A}(\mathbf{u})(x) = -\mathbf{div}\,\widehat{\mathbf{T}}(x, \mathbf{I} + \nabla\mathbf{u}(x)), \qquad x \in \Omega,$$

maps the Sobolev space $\mathbf{W}^{2,p}(\Omega)$ into the space $\mathbf{L}^p(\Omega)$ for each $p > 3$, and further, it is Fréchet-differentiable between these same spaces. These results, which are due to Valent [20], rely essentially on the fact that the Sobolev space $W^{1,p}(\Omega)$ is an algebra for $p > 3$.

Hence one way of solving the pure displacement problem (26) consists in finding

$$(28) \qquad \mathbf{u} \in \mathbf{V}^p(\Omega) = \{\mathbf{v} \in \mathbf{W}^{2,p}(\Omega);\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma\}, \qquad p > 3,$$

such that

$$(29) \qquad \mathbf{A}(\mathbf{u}) = \mathbf{f}.$$

In order to use the implicit function theorem in a neighborhood of the origin, we must verify that *the Fréchet derivative* $\mathbf{A}'(\mathbf{0})$ *is an isomorphism between the spaces* $\mathbf{V}^p(\Omega)$ *and* $\mathbf{L}^p(\Omega)$.

But the equation $\mathbf{A}'(\mathbf{0})\mathbf{u} = \mathbf{f}$ is precisely a *boundary value problem of linearized elasticity.* If we make the additional assumptions that the (frame-indifferent) material is homogeneous (its response function is independent of $x \in \overline{\Omega}$) and isotropic ("at any point in the reference configuration, its response is the same in all directions"), the equation $\mathbf{A}'(\mathbf{0})\mathbf{u} = \mathbf{f}$ takes the familiar form

$$(30) \qquad \begin{cases} -\mathbf{div}\{\lambda(\mathrm{tr}\,\mathbf{e}(\mathbf{u}))\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})\} = \mathbf{f} \\ \quad \text{in } \Omega, \text{ with } \mathbf{e}(\mathbf{u}) = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \end{cases}$$

where λ and μ are two constants, called the *Lamé constants* of the material. Experimental evidence shows that, for actual elastic materials, one has

$$(31) \qquad \lambda > 0, \qquad \mu > 0.$$

It is well known that, under the assumptions (31) (actually the inequalities $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$ would suffice) the linear problem (30) possess a unique weak solution in the space $\mathbf{H}_0^1(\Omega)$ (this relies crucially on *Korn's inequality).* It can be further shown that, if the boundary Γ is smooth enough, the continuous, injective, operator $\mathbf{A}'(\mathbf{0}) : \mathbf{V}^p(\Omega) \to \mathbf{L}^p(\Omega)$ is also surjective, i.e. that a *regularity result* of the form

$$\mathbf{A}'(\mathbf{0})\mathbf{u} \in \mathbf{L}^p(\Omega) \Rightarrow \mathbf{u} \in \mathbf{V}^p(\Omega)$$

holds. Combining this regularity result with the implicit function theorem, we obtain a *"local" existence theorem in the space* $\mathbf{W}^{2,p}(\Omega)$, $p > 3$: *For each number* $p > 3$, *there exist a neighborhood* $\mathbf{F}^p$ *of* $\mathbf{0}$ in $\mathbf{L}^p(\Omega)$ *and a neighborhood* $\mathbf{U}^p$ *of* $\mathbf{0}$ *in* $\mathbf{V}^p(\Omega)$ *such that, for each* $\mathbf{f} \in \mathbf{F}^p$, *the pure displacement problem* (30) *has exactly one solution* $\mathbf{u} \in \mathbf{U}^p$.

Through a possible reduction of the neighborhood $\mathbf{F}^p$, one can further show that the associated mapping $\boldsymbol{\varphi} = \mathbf{id} + \mathbf{u}$ is a deformation, i.e. that it is *orientation-preserving* and *injective* on $\overline{\Omega}$ (the injectivity, which relies on properties of the topological degree, may be proved as in Ciarlet [8, Theorem 5.5–2]).

The successful application of the implicit function theorem to existence theory thus relies on two keystones:

(i) *the differentiability of the operator* $\mathbf{A}$ (cf. (27)) of nonlinear elasticity between the spaces $\mathbf{W}^{2,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$;

(ii) *the surjectivity of the derivative* $\mathbf{A}'(\mathbf{0})$, or equivalently, the regularity property that the weak solution of the linearized problem (this solution is known to exist "at least" in the space $\mathbf{H}^1(\Omega)$ by the variational theory) lies in the space $\mathbf{W}^{2,p}(\Omega)$ if the right-hand side $\mathbf{f}$ is in $\mathbf{L}^p(\Omega)$.

The extensions of this approach are thus limited to situations where both properties still hold. For instance, analogous existence results hold for pure traction problems if the boundary Γ is smooth enough (specific difficulties arise however in this case, which thus requires special care; see in particular Chillingworth, Marsden & Wan [6, 7], Valent [21], Le Dret [15]).

It is the lack of $\mathbf{W}^{2,p}(\Omega)$*-regularity of the solutions of linearized displacement-traction problems* (except in the very special case where $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$) *that prevents the application of the implicit function theorem to genuine displacement-traction problems.* To overcome this lack of regularity, one could conceivably try to apply the implicit function theorem in Sobolev spaces "of lower order" such that $\mathbf{H}^1(\Omega)$, where the solution of the linearized problem is known to lie anyway.

This idea cannot be pursued any further, however, even in the case of pure displacement problems (hence let alone in the case of displacement-traction problems!), in view of the following striking *result of nondifferentiability*, due to Valent and Zampieri [22]: Assume that the operator $\mathbf{A}$ is a homomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}_0^{1,p}(\Omega)$ and a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{-1,p'}(\Omega)$ (the dual of $\mathbf{W}_0^{1,p}(\Omega)$) for some $p > 1$. Then the response $\widehat{\mathbf{T}}$ is necessarily affine! In other words, any nonlinear operator is ruled out by this approach, and we know that $\widehat{\mathbf{T}}$ cannot be linear when the reference configuration is a natural state (cf. §2).

This is an instance of the nondifferentiability of *Nemytsky operators*, also called substitution, or composition, operators, which are studied at length by T. Valent in his book.

§6. The book of T. Valent

The book under review is an up-to-date, complete, and self-contained, exposition of the "local" existence theory in three-dimensional elasticity based on the implicit function theorem.

After a brief survey of three-dimensional elasticity, the author gives a detailed treatment of the differentiability or nondifferentiability, and analyticity, of nonlinear operators acting between Sobolev spaces $\mathbf{W}^{m,p}(\Omega)$, or between Schauder spaces $\mathbf{C}^{m,\lambda}(\overline{\Omega})$. He also gives a detailed treatment of the existence and regularity of solutions of problems in linearized elasticity. As we have seen (cf. §5), these questions play a crucial rôle in the application of the implicit function theorem to existence theory in elasticity.

Local existence theorems, as well as uniqueness, analytic dependence on the right-hand sides, topological properties of the sets of admissible deformations, are then proved in a variety of situations that include pure displacement problems and pure traction problems, in particular for dead loads and pressure loads. All these make a wholeheartedly recommended supplementary reading to

Chapter 6 of Ciarlet [8], where this approach (including a proof of convergence of an incremental method) is also exposed in detail, but only for the pure displacement problem.

The treatment is mathematically rigorous and thorough. The book is well, and carefully, written (however, many statements of theorems, corollaries, or lemmas could have been shortened!); it is complete, with historical notes, an index of notations, and an index. Finally, the typesetting and general outlook are in the best Springer tradition.

I strongly recommend this book to anyone interested by the modern mathematical theory of elasticity.

References

1. S. S. Antman, *Existence of solutions of the equilibrium equations for nonlinearly elastic rings and arches*, Indiana Univ. Math. J. **20** (1970), 281–302.
2. ——, *Regular and singular problems for large elastic deformations of tubes, wedges, and cylinders*, Arch. Rational. Mech. Anal. **83** (1983), 1–52.
3. J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.
4. ——, *Global invertibility of Sobolev functions and the interpenetration of matter*, Proc. Roy. Soc. Edinburgh **88A** (1981), 315–328.
5. M. van Buren, *On the existence and uniqueness of solutions to boundary value problems in finite elasticity*, Thesis, Carnegie-Mellon University, Pittsburgh, 1968.
6. D. R. J. Chillingworth and J. E. Marsden, *Symmetry and bifurcation in three-dimensional elasticity, Part* I, Arch. Rational Mech. Anal. **80** (1982), 295–331.
7. ——, *Symmetry and bifurcation in three-dimensional elasticity, Part* II, Arch. Rational Mech. Anal. **83** (1983), 363–395.
8. P. G. Ciarlet, *Mathematical elasticity, Vol.* I: *Three-dimensional elasticity*, North-Holland, Amsterdam, 1988.
9. P. G. Ciarlet and P. Destuynder, *A justification of a nonlinear model in plate theory*, Comput. Methods Appl. Mech. Engrg. **17/18** (1979), 227–258.
10. P. G. Ciarlet and J. Nečas, *Unilateral problems in nonlinear, three-dimensional elasticity*, Arch. Rational Mech. Anal. **87** (1985), 319–338.
11. ——, *Injectivity and self-contact in nonlinear elasticity*, Arch Rational Mech. Anal. **19** (1987), 171–188.
12. B. D. Coleman and W. Noll, *On the thermostatics of continuous media*, Arch. Rational Mech. Anal. **4** (1959), 97–128.
13. R. L. Fosdick and J. Serrin, *On the impossibility of linear Cauchy and Piola-Kirchhoff constitutive theories for stress in solids*, J. Elasticity **9** (1979), 83–89.
14. M. E. Gurtin, *In introduction to continuum mechanics*, Academic Press, New York, 1981.

15. H. Le Dret, *Structure of the set of equilibrated loads in nonlinear elasticity and applications to existence and nonexistence*, J. Elasticity **17** (1987), 123–141.

16. J. E. Marsden and T. J. R. Hughes, *Topics in the mathematical foundations of elasticity*, in Nonlinear analysis and mechanics: Heriot-Watt symposium, Vol. 2, Pitman, London, 1978, pp. 30–285.

17. ——, *Mathematical foundations of elasticity*, Prentice-Hall, Englewood Cliffs, 1983.

18, F. Stoppelli, *Un teorema di esistenza e di unicità relativo alle equazioni dell' elastostatica isoterma per deformazioni finite*, Ricerche Mat. **3** (1954), 247–267.

19. C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, in Handbuch der Physik, Vol. III/3 (S. Flügge, Editor), Springer-Verlag, Berlin, 1965, pp. 1–602.

20. T. Valent, *Local theorems of existence and uniqueness in finite elastostatics*, in Finite elasticity (D. E. Carlson and R. T. Shield, Editors), Nijhoff, The Hague, 1978, pp. 401–421.

21. ——, *Pressure boundary problems in finite elasticity. Results on local existence, uniqueness and analyticity*, in Proceedings, meeting on finite thermoelasticity (G. Grioli, Editor), Accademia Nazionale dei Lincei, Roma, 1986, pp. 241–264.

22. T. Valent and G. Zampieri, *Sulla differenziabilità di un operatore legato a una classe di sistemi differenziali quasi-lineari*, Rend. Sem. Mat. Univ. Padova **57** (1977), 311–322.

23. C.-C. Wang and C. Truesdell, *Introduction to rational elasticity*, Noordhoff, Groningen, 1973.

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Heat kernels and spectral theory, by E. B. Davies. Cambridge University Press, Cambridge, 1989, 197 pp., $49.50. ISBN 0-521-36136-2

For the usual Laplacian Δ in Euclidean space $\mathbf{R}^n$ and its spectral theory, there is a tremendous amount of information available, largely because of a number of explicit formulas that are known. For example, the heat semigroup $e^{t\Delta}$ (the operator that solves the heat equation $\partial u/\partial t = \Delta u$ for $t > 0$ from the initial value $u(x, 0)$, given certain weak growth conditions) is a convolution operator with kernel $(4\pi t)^{-n/2}e^{-|x|^2/4t}$. From the