BOOK REVIEWS

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Mathematical intuitionism. Introduction to proof theory, by A. G. Dragalin. Translations of Mathematical Monographs, Vol. 67. Translated by E. Mendelson. American Mathematical Society, Providence, R. I., 1988, ix+228 pp., $75.00. ISBN 0-8218-4520-9

A common misconception among mathematicians is to think of intuitionistic mathematics as "mathematics without the law of the excluded middle" (the law asserting that every statement is either true or false). From this point of view, intuitionistic mathematics is a proper subset of ordinary mathematics, and doing your mathematics intuitionistically is like doing it with your hands tied behind your back.

Another more realistic viewpoint is to regard intuitionistic logic, and the mathematics based on that logic, as the logic of sets with some structure, rather than of bare sets. Traditional examples are sets growing in time (as in Kripke semantics [9]), or set with some recursive structure (as in Kleene's realizability interpretation [7]), or sets continuously varying over some fixed parameter space. Universes of such sets are perfectly suitable for developing mathematics, but one is often *forced* to use intuitionistic logic.

This is a small price to pay for the many new phenomena that can be observed in such universes: For example, in some such universes one has a "recursive axiom of choice," which states that for any sequence $\{A_n\}_n$ of nonempty subsets $A_n \subset \mathbf{N}$ there is a *recursive* choice function $f: \mathbf{N} \to \mathbf{N}$ selecting an element $f(n)$ from each set; A_n in particular, *any* function from the natural numbers to itself is recursive. (This property is known as "Church's thesis.") In other universes, one can prove "Brouwer's theorem", which states that all functions from the reals to the reals are continuous. And as a third example, there are universes in which one can model *exactly* the arguments of geometers like S. Lie and E. Cartan, who used nilpotent infinitesimal numbers in a way *incompatible* with classical logic. (See [8, 10]; caveat: this has nothing to do with nonstandard analysis [12], which only deals with invertible infinitesimals.)

In the first half of this century, the interest in intuitionistic logic and mathematics was mainly of a philosophical and foundational nature. More recently, it has become apparent that intuitionistic logic or some variant thereof is often the right logic to use in theories of computing. And intuitionistic logic has been shown to be more intimately connected to "mainstream mathematics" through the stunning discovery by F. W. Lawvere and M. Tierney that Grothendieck's sheaf theory and intuitionistic set theory are essentially the same thing.

I should mention one aspect of the correspondence between sheaves and logic in more detail. Let $\mathbf{C}$ be a category equipped with the notion of "covering families" (for example, $\mathbf{C}$ may be the poset of open subsets of a fixed topological space, viewed as a category, with the usual notion of covering families). Grothendieck associates with such data the category $\mathrm{Sh}(\mathbf{C})$ of sheaves on $\mathbf{C}$. This category is a model for intuitionistic set theory, and the truth of set-theoretic formulas in this model can be described in terms of a forcing relation $C \Vdash \varphi(x_1, \dots, x_n)$ where φ is such a formula and C is an object from $\mathbf{C}$. Traditional models for intuitionistic logic, such as those of Kripke [9] and Beth [1] are special cases of these general sheaf models, as are Cohen's models for proving the independence of the continuum hypothesis [2, 13]. Futhermore, one of Verdier's gros topos [5] is a model for "Brouwer's theorem" mentioned above, while a "smooth" analogue of the Zariski topos models the infinitesimals of Lie and Cartan mentioned above [3, 11]. Universes of "sets with recursive structure" also fit into this context [6].

When asked to review the book by Dragalin, I faced the problem whether to review the original text in Russian or the AMS translation. Not that I am able to read Russian, but most of the developments concerning intuitionism and sheaves did not become widely known before the late seventies, while the Russian text appeared in 1979, its translation in 1988. As a consequence, the English edition of Dragalin's book is a rather traditional introduction to intuitionistic logic, and — unlike other contemporary introductions — contains nothing above sheaf semantics.

The core of Dragalin's book consists of four chapters. The first chapter discusses the proof theory of intuitionistic first-order logic, and basic properties like the disjunction and existence properties are derived. The second chapter focusses on intuitionistic arithmetic and Kleene's realizability interpretation. The third chapter discusses model theory for intuitionistic logic, and the fourth deals with intuitionistic analysis, in particular various axiom systems for choice sequences. (There are also a chapter on cut elimination and two appendices, all of more special interest.)

These four chapters occupy only 150 pages, and form a remarkably quick introduction to the subject. I consider this one of the main positive aspects of this book. Especially the second chapter is a nice introduction to realizability, covering many of the basic properties. But brevity has its price, and there is very little in the book in terms of motivation or examples.

As said, the third chapter on semantics does not use some of the more recent insights and simplifications. It unfortunately does not provide the reader with sufficient background to study some of the more advanced literature, and he or she may prefer to consult the expository paper [4] or the relevant parts of [14].

The translation, by E. Mendelson, seems carefully done (although the terminology is sometimes slightly nonstandard; e.g., the disjunction property is referred to as "the disjunctiveness property"), and there are very few misprints. But more than a century after Brouwer's birth, the spelling of "intuitionistic" still gives typesetters headaches (cf. p. 23).

REFERENCES

1. E. W. Beth, *Semantic construction of intuitionistic logic*, Kon. Ned. Ak. Wet. **19** (1956), 357–388.
2. P. J. Cohen, *Set theory and the continuum hypothesis*, Benjamin, New York, 1966.
3. E. Dubuc, *C^∞-schemes*, Amer. J. Math **103** (1981).

4. M. Fourman and D. Scott, *Sheaves and logic*, in Application of Sheaves, Springer LNM **753**, 1979, pp. 280–401.
5. A. Grothendieck, et al., *Theorie des topos et cohomologie étale des schémes*, Springer LNM **269**, 1972.
6. J. M. E. Hyland, *The effective topos*, The L. E. J. Brouwer Centenary Sympos. (A. S. Troelstra, D. van Dalen, eds.), North-Holland, Amsterdam, 1982.
7. S. C. Kleene, *On the interpretation of intuitionistic number theory*, J. Symbolic Logic **10** (1945), 109–124.
8. A. Kock, *Synthetic differential geometry*, Cambridge Univ. Press, Cambridge, United Kingdom, 1981.
9. S. Kripke, *Semantical analysis of intuitionistic logic, I*, in Formal Systems and Recursive Functions (J. Crossely and M. A. E. Dummett, eds.), North-Holland, Amsterdam, 1965, pp. 92–130.
10. R. Lavendhomme, *Leçons de géométrie différentielle synthétique naïve*, CIACO, Louvain-la-Neuve, 1987.
11. I. Moerdijk and G. E. Reyes, *A smooth version of the Zariski topos*, Adv. Math. **65** (1987), 229–253.
12. A. Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.
13. M. Tierney, *Sheaf theory and the continuum hypothesis*, in Toposes, Algebraic Geometry and Logic (F. W. Lawvere, ed.), Springer LNM **274**, 1972.
14. A. S. Troelstra and D. van Dalen, *Constructivism in mathematics*, vol. **II**, North-Holland, Amsterdam, 1988.

IEKE MOERDIJK
MATHEMATISCH INSTITUUT
BUDAPESTLAAN 6
POSTBUS 800010
3508 TA UTRECHT
THE NETHERLANDS

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