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A SYMPLECTIC GEOMETRY APPROACH TO GENERALIZED CASSON'S INVARIANTS OF 3-MANIFOLDS

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1. In 1985 lectures at MSRI, Andrew Casson introduced an integer valued invariant $\lambda(M)$ for any oriented integral homology 3-sphere M^3. This invariant has many remarkable properties; detailed discussions of some of these can be found in an exposé by S. Akbulut and J. McCarthy (see [AM]). Roughly, $\lambda(M)$ measures the 'oriented' number[3] of irreducible representations of the fundamental group $\pi_1(M)$ in $SU(2)$.

In the preceding article of this journal, Kevin Walker [W] described results of his thesis which yield an invariant $\lambda(M^3)$ of an arbitrary oriented rational homology 3-sphere (RHS: $H_1(M^3, Q) = 0$) which extends Casson's invariant. His creative methods give generalizations of the properties which Casson had earlier shown for oriented integral homology 3-spheres (IHS: $H_1(M^3, \mathbf{Z}) = 0$). For homology lens spaces, Boyer and Lions [BL] have independently obtained an inductive definition of $\lambda(M^3)$. Earlier, a different extension of Casson's invariant to certain rational homology spheres, which does not equal Walker's invariant, had been studied by S. Boyer and A. Nicas [BN]. In all of the above works, one is considering only representations into $SU(2)$.

The present announcement solves the problem, which has been emphasized by Atiyah [A], of producing generalizations of Casson's invariant to invariants of M^3 that would roughly measure the 'oriented' number of representations of $\pi_1(M)$ in $G = SU(n)$, for each $n \geq 2$. We introduce $\lambda_G(M^3)$, an invariant which is defined for an arbitrary oriented rational homology 3-sphere (RHS).

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[3] Following [W] we do not divide by 2 as in [A-M].

These results work more generally for arbitrary semisimple Lie groups G. As expected, $\lambda_{SU(2)}(M^3)$ equals Walker's $\lambda(M^3)$ when M^3 is a RHS. The present definition of $\lambda_{SU(n)}(M^3)$ for M^3 an integral homology 3-sphere (IHS) draws only on some of Casson's original ideas, some notions of symplectic geometry, and results of Bismut and Freed [BF] here adapted to stratified varieties. Our further extension to M^3 a RHS for $SU(n)$ uses also the methods introduced by Walker to solve the analogous problem for $SU(2)$.

2. Given a compact Lie group G and a finite CW complex X, we denote by $\widehat{R}_G(X)$ the space of representations of $\pi_1(X)$ into G, and by $R_G(X)$ the space of conjugacy classes of representations, i.e., $\widehat{R}_G(X) = \mathrm{Hom}(\pi_1(X), G)$ and $R_G(X) = \mathrm{Hom}(\pi_1(X), G)/G$. In the case of an oriented 3-manifold M^3, the above spaces $R_G(M)$ and $\widehat{R}_G(M)$ can be studied by means of a Heegaard decomposition of M. Let $M = W_1 \bigcup_F W_2$ where W_1 and W_2 are handle bodies with g handles, and $F = W_1 \cap W_2 = \partial W_1 = \partial W_2$, a Riemann surface of genus g. Then the push-out diagram of surjections of fundamental groups

$$\text{(2.1)} \qquad \begin{tikzcd} & \pi_i(W_2) \arrow[dr] & \\ \pi_1(F) \arrow[ur] \arrow[dr] & & \pi_1(M^3) \\ & \pi_1(W_1) \arrow[ur] & \end{tikzcd}$$

gives rise to a corresponding diagram of inclusions of representation spaces

$$\text{(2.2)} \qquad \begin{tikzcd} & R_G(W_1) \arrow[dl] & \\ R_G(F) & & R_G(M) ; \arrow[ul] \arrow[dl] \\ & R_G(W_2) \arrow[ul] & \end{tikzcd}$$

in particular, $R_G(M) = R_G(W_1) \cap R_G(W_2)$. There is a similar diagram for the corresponding G-spaces $\widehat{R}_G(M)$, $\widehat{R}_G(W_1)$, $\widehat{R}_G(W_2)$, and $\widehat{R}_G(F)$.

As in [G] and [K], the space $R_G(F)$ has the structure of a complex variety whose nonsingular part $R_G(F)_{\text{irred.}}$ consists of precisely the equivalence classes of irreducible representations $\pi_1(F) \to G$. $R_G(W_1)$ and $R_G(W_2)$ each have half of the dimension of the ambient space $R_G(F)$. Hence, when the intersection $R_G(M)\cap R_G(F)_{\text{irred.}} = R_G(W_1)\cap R_G(W_2)\cap R_G(F)_{\text{irred.}}$ can be separated from the singular part $R_G(F)_{\text{red.}}$ of $R_G(F)$, we can put $R_G(W_1)_{\text{irred.}}$ and $R_G(W_2)_{\text{irred.}}$ into general position to get an intersection number $\langle R_G(W_1)_{\text{irred.}}, R_G(W_2)_{\text{irred.}}\rangle$. Following Casson, it is natural to consider the expression

$$(2.3)\qquad (-1)^{G(g)}(1/T)\langle R_G(W_1)_{\text{irred.}}, R_G(W_2)_{\text{irred.}}\rangle$$

where T is the order of the torsion of $H_1(M^3, 8)$, $R_G(W_1)$, $R_G(W_2)$ are compatibly oriented as in [J], and $e(g) = (\dim G)(g(g-1)/2)$. These compatible orientations are achieved without assuming M^3 is a RHS. When $G = SU(2)$ and M is an IHS, the intersection $R_G(W_1)_{\text{irred.}} \cap R_G(W_2)_{\text{irred.}}$ is separated from the trivial representation—the only other representation in $R_G(W_1)\cap R_G(W_2)$. The expression (2.3) equals Casson's invariant in this case. For general G, or even for $G = SU(2)$ and M an RHS as in [W], the separation condition fails and the value of the expression (2.3) will depend upon how the $R_G(W_j)$, $j = 1, 2$ are placed into general position. Consequently, we will describe how to modify (2.3) to obtain a well-defined invariant.

3. We first review the stratifications of the representation spaces and the corresponding symplectic geometry. Let $\rho\colon \pi_1(F) \to SU(n)$ be a representation. Then ρ can be written as a sum of irreducible representations ρ_i, $\rho = \oplus_{i=1}^{b} m_i\rho_i$, and in this way we can associate to ρ a finite set of pairs of integers (m_i, n_i), $i = 1, \dots, b$, where m_i is the multiplicity of ρ_i in ρ and $n_i = \dim\rho_i$. Clearly we have

$$(3.1)\qquad \sum_{i=1}^{b} m_i n_i = n,$$

and if we are given such a finite set $\beta = \{(m_i, n_i)\}_{i=1}^{b}$ of pairs of positive integers satisfying (3.1), then there exists a subspace S_β in $R_G(F)$ consisting of all representations ρ with decomposition $\rho = \oplus_{i=1}^{b} m_i\rho_i$, ρ_i irreducible, $n_i = \dim\rho_i$. There is a partial ordering among these indexing sets $\{\beta\}$ according to the closure relation of the corresponding subspaces S_β. In this way, the space $R_{SU(n)}(F)$ has the structure of an orbifold stratification $\{S_\beta\}$ (see [K]). The word "orbifold" refers to the fact S_β's are not manifolds but orbifolds, and in fact given the data on β as before, S_β can be identified with the orbit space

$$(3.2)\qquad S_\beta = \frac{\operatorname{Hom}(\pi_1(F)\,,\, Su(n)\cap U(n_1\,,n_2\,,\ldots\,,n_b))_{\text{irred.}}}{SU(n)\cap\{\mathfrak{S}\ltimes U(n_1\,,n_2\,,\ldots\,,n_b)\}}$$

where $U(n_1\,,n_2\,,\ldots\,,n_b)$ is the product of unitary groups $U(n_1)\times U(n_2)\times\cdots\times U(n_b)$ and $\mathfrak{S}$ is the product of symmetric groups $\prod_{m\geq 0\,,n\geq 0}\mathfrak{S}\ \ (\#[j|m_j=m\,,\ n_j=n\})$.

In addition to the orbifold stratification, the space $R_G(F)$ has a ambient symplectic structure, and each stratum S_β has a compatible symplectic orbifold structure. More precisely, let ρ be given as before, and let $\operatorname{ad}\rho = \operatorname{ad}(\oplus m_i\rho_i)$ denote the adjoint representation. Then the cohomology $H^1(F\,,\operatorname{ad}\rho)$ of F with coefficients in $\operatorname{ad}\rho$ has a skew symmetric pairing $\langle\ ,\ \rangle\colon H^1(F\,,\operatorname{ad}\rho)\times H^1(F\,,\operatorname{ad}\rho)\to\mathbf{R}$, given by a combination of the cup product and the Killing form on the Lie algebra $\mathbf{su}(n)$. Also if K_ρ is the isotropy subgroup ρ, then K_ρ operates on $H^1(F\,,\operatorname{ad}\rho)$ preserving this symplectic pairing.

As is well known in symplectic geometry, such a triple $(H^1(F\,,\operatorname{ad}\rho)\,,\ K_\rho\,,\ \langle\,\rangle)$, determines a well-defined moment map

$$(3.3)\qquad \mu\colon H^1(F\,,\operatorname{ad}\rho)\to k_\rho^*\left(\cong\left[\bigoplus_{i=1}^{k}\mathbf{u}(m_i)\right]\cap\mathbf{su}\left(\sum m_i\right)\right)$$

of $H^1(F\,,\operatorname{ad}\rho)$ to the dual Lie algebra k_ρ^* of K_ρ. Moreover, there exists a local homeomorphism between a neighborhood of $[\rho]$ in the representation space $R_G(F)$ and the quotient space $\mu^{-1}(0)/K_\rho$ of the inverse image $\mu^{-1}(0)\subset H^1(F\,,\operatorname{ad}\rho)$ modulo the action of K_ρ. In this local model, there is a symplectic subspace $\oplus_{j=1}^{b}H^1(F\,,\operatorname{ad}\rho_j)$ in $\mu^{-1}(0)$ with a K_ρ-action, and this corresponds to the orbifold tangent space of S_β. For these facts see [AMM] or the elegant discussion of [G]. For background see [AB].

The same analysis can be carried out for the subspace $R_G(W_j)$, $j=1\,,2$. In this case, the cohomology $H^1(W_j\,,\operatorname{ad}\rho)$ represents a K_ρ-invariant, Lagrangian subspace in $H^1(F\,,\operatorname{ad}\rho)$. It follows that, compatible with the local homeomorphism in the previous paragraph, there exists a local homeomorphism from a neighborhood of $[\rho]$ in $R_G(W_j)$ to the quotient space $H^1(W_j\,,\operatorname{ad}\rho)/K_\rho$ in $\mu_j^{-1}(0)/K_\rho$.

4. Now, to generalize Casson's invariant, our strategy is to begin by perturbing, in a strata-preserving way, the two subspaces $R_G(W_1)$ and $R_G(W_2)$ so that they intersect "transversely" at a finite number of points. As explained above, in each of the strata S_β, the subspaces $R_G(W_1)\cap S_\beta$ and $R_G(W_2)\cap S_\beta$ are Lagrangian orbifolds, and therefore we can apply the theories of Gromov and Lees (see [L] and [E]) to deform these submanifolds by regular homotopy of Lagrangian immersions. To be precise, because these are only orbifolds, we have to appeal to the equivariant extension of this immersion theory due to Bierstone (see [B 1]). In a similar manner, we can deform these orbifolds so that they become "equivariantly transverse" to each other at a finite number of points in the sense of Bierstone (see [B 2]). Furthermore, for the present application when M^3 is not an IHS the perturbation in S_β must be done while fixing its image under the (orbifold) Jacobian map.

Note, however, that the resulting number of intersection points in $R_G(F)_{\text{irred.}}$ counted with sign $\langle R_G(W_1)_{\text{irred.}}, R_G(W_2)_{\text{irred.}}\rangle$, does depend on the choices of deformations. More precisely, as we deform the Lagrangian subspaces $R(W_j)$, irreducible intersection points may be destroyed or created along the reducible strata (see [W]). To get an invariant independent of perturbations, we correct $\langle R_G(W_1)_{\text{irred.}}, R_G(W_2)_{\text{irred.}}\rangle$ by adding $\sum_P I(P)$ where P ranges over the reducible intersection points of these deformed pseudomanifolds in transverse position and $I(P)$ is an invariant computed from certain Maslov indices.

For example, suppose M is an IHS, and $G = SU(n)$; let $P = [\rho]$, the equivalence class of a reducible representation ρ with a decomposition into irreducible representations $\rho = \rho_1 \oplus \rho_2$, $\rho_j\colon \pi_1(M) \to SU(n_j)$, in a stratum of $R_{SU(n)}(F)_{\text{red.}}$. We fix a metric on F. Over the space $R_{SU(n_1)}(F)_{\text{irred.}} \times R_{SU(n_2)}(F)_{\text{irred.}}$, which maps onto the stratum containing P in $R_{SU(n)}(F)_{\text{red.}}$, there are three determinant line bundles:

$$\begin{aligned}
\mathscr{L} &= \det\nolimits_{\mathbf{C}} \not{\partial} \otimes \left(\rho_1^* \otimes \rho_2\right), \\
\mathscr{L}_1 &= \det \not{\partial} \otimes \operatorname{Ad} \rho_1, \\
\mathscr{L}_2 &= \det \not{\partial} \otimes \operatorname{Ad} \rho_2,
\end{aligned}$$

with Bismut–Freed connections (see [BF]). Based on the work of Bismut–Freed, a direct computation of the first Chern forms shows that for certain nonzero integers a, b, c, we have an *explicit* trivialization of

$$\mathscr{L}_1^a \otimes \mathscr{L}_2^b \otimes \mathscr{L}^c \cong \varepsilon . \tag{4.1}$$

This phenomenon is also known as anomaly cancellation. Over $R_{SU(n_1)}(W_j) \times R_{SU(n_2)}(W_j)$, the above three bundles also inherit canonical sections: The bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ are the determinants of the tangent bundles along $R_{SU(n_1)}(F)_{\text{irred.}}$ and $R_{SU(n_2)}(F)_{\text{irred.}}$; $\mathscr{L}$ is the determinant of the bundle associated to the normal cone of this stratum. The Lagrangian nature of $R_{SU(n_1)}(W_j) \times R_{SU(n_2)}(W_j)$, and of $R_{SU(n)}(W_j)$ provides the desired sections of $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}$, and thus a section s of $\mathscr{L}_1^a \otimes \mathscr{L}_2^b \otimes \mathscr{L}^c$. The data of bundles, sections, and trivializations extend to the closure of the strata.

Next we choose paths γ_j from the trivial representation [1] to $P = [\rho]$ in $R_{SU(n_1)}(W_j) \times R_{SU(2)}(W_j)$, $j = 1, 2$. The three sections of $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}$ over the loop $\gamma = \gamma_1 * \gamma_2^{-1}$ (along γ_1 then along γ_2 reversed) provide a section of the trivial bundle (4.1) and so a winding number called *the Maslov index* of the point P. See e.g. [V]. In this case, $I(P)$ is a certain multiple of this Maslov index; this multiple depends only on n_1, n_2, and the sign of the intersection of $R_{SU(n_1)}(W_j) \times R_{SU(n_2)}(W_j)$, $j = 1, 2$ in $R_{SU(n_1)}(F) \times R_{SU(n_2)}(F)$ at $[\rho_1 \otimes \rho_2]$. More generally, if M is an IHS and P lies in a higher stratum, one must form several Maslov indices and $I(P)$ is expressed as a polynomial in these indices.

The above Maslov index has an alternative description as $\int_\gamma s^*(\theta)$, where θ is the connection 1-form on $\mathscr{L}_1^a \otimes \mathscr{L}_2^b \otimes \mathscr{L}^c$ which comes from the canonical Bismut–Freed connections on $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}$. More generally, when M^3 is a RHS we can use this integral in place of the Maslov index used above in defining $I(P)$. For $G = SU(2)$ and $P = [\rho]$ with ρ reducible, this agrees with Walker's $I(P)$, which he defined using the Kähler form.

In a way, the invariant $\lambda_G(M)$ is "topological" in the sense of Witten [W]; in other words, it is invariant under the Lagrangian perturbations, though one must be careful to control them as above.

Casson showed that for M^3 an ISH, and, more generally, Walker, for M^3 an oriented RHS, $\lambda_G(-M) = -\lambda_G(M)$, for $G = SU(2)$, where $-M$ denotes M with the opposite orientation. This generalizes to:

Theorem. $\lambda_G(-M) = (-1)^{\dim G}\lambda_G(M)$ *for* M^3 *an oriented* RHS.

Further results, detailed proofs, and discussion will appear elsewhere.

Added in proof: Using the notion of clean intersections, the invariant $\lambda_G(M)$ may be defined more generally for M^3 any closed oriented 3-manifold.

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