relation B satisfies five axioms, one of which is Pasch's Law, and the other is a version of Dedekind continuity.

Chapter 3, *Projective transformations* "develops an alternative method of coordinatising a metric affine plane by embedding it into a projective plane, and using the orthogonality relation to define a matrix-representable transformation on the line at infinity. The construction will be central to the subsequent treatment of metric affine spaces of higher dimension."

The investigations conducted by the author in Chapters 3 and 4 show that there exist only two nonsingular metric affine threefolds (Euclidean and Minkowskian spaces) and only three nonsingular metric affine fourfolds (Euclidean, Artian and Minkowskian spaces), if these spaces carry the ternary relation "between." Such spaces are called continuously ordered.

The word "order" occurs in Appendix B "After and the Alexandrov-Zeeman Theorem" for the second time, where the author shows that the notions "between" and "orthogonal" can be defined in terms of the notion of "after", i.e. an ordering which is given on affine space. Hence the method of axiomatization of spacetime based on the orthogonality relation must be considered as part of the program of the construction of causal theory of spacetime which was proposed by A. D. Alexandrov [**1**, **2**].

This book is read with pleasure, and will be useful for the students who are wishing to become geometers.

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Partially ordered abelian groups with interpolation, by K. R. Goodearl. Mathematical Surveys and Monographs, number 20, American Mathematical Society, Providence, R.I., 1986, xxii + 336 pp., ISBN 0-8218-1520-2

In [**19**], F. Riesz introduced what has come to be called the Riesz decomposition property. An ordered (abelian) group is said to have the Riesz decomposition property if the sum of two order intervals is again an order interval. Riesz showed, among other things, that the cone of positive additive functionals on an ordered group with this property is a lattice. (In the case that the group has an order unit, this says that the compact convex set of positive additive functionals equal to one on the order unit is a simplex.)

In [1], G. Birkhoff showed that the Riesz decomposition property is equivalent to what he called the Riesz interpolation property, a property which is expressed purely in terms of the order structure of the ordered group: if x_1, x_2 and y_1, y_2 are elements with $x_i \leq y_j$ for all i and j, then there exists an element z with $x_i \leq z \leq y_j$ for all i and j. This property, of interest in itself, is often more convenient to verify in an ordered group than the Riesz decomposition property. For instance, Riesz proved that an ordered group which is a lattice has the decomposition property,—but it can be seen immediately that any lattice has the Riesz interpolation property.

As examples of lattice-ordered groups,—besides the span of the positive functionals on an ordered group with the interpolation property—, Riesz pointed out the group of continuous functions on a topological space, and the group of harmonic functions in the plane, both with the pointwise order.

As an example of an ordered group with the interpolation property which is not a lattice, Riesz mentioned the field of rational functions on an interval. His proof of the decomposition property in this case is simple, but powerful: it is valid for any ordered field, and consists in observing that, for every $x \geq 0$, $[0, x] = x[0, 1]$, where $[0, x]$ denotes the order interval $\{a | 0 \leq a \leq x\}$. In any ordered commutative ring in which this property holds, the Riesz decomposition property (which says that $[0, a+b] = [0, a] + [0, b]$ whenever $a, b \geq 0$) follows immediately (by setting $x = a + b$). Another example (besides a field) of an ordered ring with the property $[0, x] = x[0, 1]$, $x \geq 0$, is the ring of real-analytic functions on an open interval, with the pointwise order.

Inspection of Riesz's proof reveals the following, related, result. If an ordered commutative ring has the property that, for every $x \geq 0$, $[0, x^2] = x[0, x]$ and, moreover, the map $[0, x] \ni\mapsto xa \in [0, x^2]$ is injective, then the Riesz decomposition property holds. To see this, note first that the map $[0, x] \ni a \mapsto xa \in [0, x^2]$ must be an order isomorphism. (If $a, b \in [0, x]$ and $xa \geq xb$, so that $x(a-b) \in [0, x^2]$, then $x(a-b) = xc$ with $c \in [0, x]$, and, as $a - b$ and c belong to $[-x, x] = [0, 2x] - x$, by injectivity of $[0, 2x] \ni y \mapsto 2xy$ one has $a - b = c$, i.e., $a \geq b$.) Now, if $0 \leq a, b$ and $c \in [0, a+b]$ then, with $w \in [0, a+b]$ such that $ac = w(a+b)$, one verifies that $w \in [0, a]$ and $c - w \in [0, b]$; this uses the hypothesis with $x = a + b$ (and also with $x = 2(a+b)$ and $x = 4(a+b)$). The property hypothesized above holds in the ring of C^n-functions on a manifold, with the pointwise order, exactly when $n = 0$ or $n = 1$, or else $n = \infty$ and the dimension of the manifold is one. That this ordered ring has the Riesz decomposition property in the last two cases appears not to have been noted. Whether it does in general is not clear.

Another example of an ordered ring with the Riesz decomposition property is the polynomials on an interval, with the pointwise order. That the polynomials have this property was proved by Fuchs in [11], using the Weierstrass approximation theorem. The Riesz property for polynomials on an interval in turn implies the Weierstrass approximation theorem, and a remarkable proof of the Riesz property by Renault in **[18]** (using Pascal's triangle, and obtaining a rather sharper form of the property) yields in this way a new proof of the Weierstrass theorem (by means of polynomials somewhat resembling the Bernstein polynomials).

The simplest example of an ordered group with the Riesz interpolation property must be the integers. This example is so elementary that it is not surprising that it receives no mention in the early work on the subject. Again, Fuchs in **[11]** obtains such important results concerning ideals and quotients (notably, that the order relation in a Riesz group is determined by its images in prime quotients), that one is not surprised to see no mention of ordinary direct sums. Similarly, it is, a priori, not noteworthy that the Riesz interpolation property is preserved under direct limits of ordered groups. Nevertheless, this accumulated neglect adds up to somewhat of a catastrophe,—for, as it turns out, every ordered group with the Riesz interpolation property, together with the minor additional property (considered by Fuchs) that if $nx \geq 0$ for some $n = 2, 3, \ldots$ then $x \geq 0$, can be expressed as a direct limit of finite direct sums of copies of the integers. This result was obtained in the totally ordered case by the reviewer in **[9]** (and in certain related cases in **[10]**), and was obtained in the divisible case by Shen in **[20]**; it was proved in general by Effros, Handelman, and Shen in **[6]**.

The reviewer's proof in the totally ordered case yields a sharper result (in that case),—that the sequence (or net) of finite ordered group direct sums of copies of **Z** can be chosen with injective mappings. This is not true in general—as shown by an example in **[10]**. (But it may be true in the divisible case, and Mundici has just proved it in the lattice-ordered case; see **[15]**.) The reviewer's proof in the totally ordered case,—to be specific, in the case of rank two—, amounts, in essence, to an application of the continued fraction algorithm, together with a proof (by induction) that this algorithm converges. This coincidence was pointed out by Effros and Shen in **[7]**. It follows that one may actually view this result as a reformulation of the continued fraction expansion of a real number (and use the usual convergence proof). This point of view, and, specifically, the uniqueness of the continued fraction expansion, was used by Cuntz and Krieger in **[4]** to solve an instance of the problem of Williams concerning shift equivalence of matrices with positive integral entries. Known properties of the continued fraction expansion (and its rate of convergence), interpreted from this point of view, were also used by Pimsner and Voiculescu in their embedding of the irrational rotation C^*-algebra in an AF-algebra, in **[17]**.

The basis underlying the applications just referred to is the result of the reviewer in **[8]**, classifying separable unital AF-algebras by the ordered K_0-group, together with the K_0-class of the unit element. This is also the first appearance of the last-mentioned examples of Riesz groups,—namely, **Z**, direct sums of copies of **Z**, and direct limits of such. These appear because AF-algebras are C^*-algebra direct limits of finite-dimensional C^*-algebras, which are direct sums of matrix algebras over the complex numbers, each of which has K_0-group **Z**. In terms of this classification result, the theorem of Effros, Handelman, and Shen may be thought of as giving an intrinsic description of what can arise as the invariant.

The theorem of Effros, Handelman, and Shen is of fundamental importance, both for the theory of ordered structures, and for the theory of C^*-algebras (and related areas). To choose an illustration cited in the book under review (and originating with the author), one obtains from this result that any simplex is an inverse limit of finite-dimensional simplices, a result due to Lazar and Lindenstrauss (in the metrizable case—**[14]**). As another illustration, one can use this result to construct what may be regarded as quantum statistical mechanical systems with arbitrarily specified thermodynamical equilibrium behaviour (i.e., arbitrary phase diagram structure)—see e.g. **[2]**.

The ordered groups arising in the classification of AF-algebras, and characterized by Effros, Handelman, and Shen, are often referred to as dimension groups. It seems to the reviewer that the theory of dimension groups has only begun.

The book under review provides a good introduction to the subject. This is to be expected, as the author has made major contributions to the field himself. In fact, this book is the first systematic account of the subject. Other, briefer, surveys that might be mentioned are those of Effros **[5]**, and of Vershik and Kerov, **[21]**. Recent work, which remains to be categorized, but in which dimension groups clearly play a rôle, is that of Handelman on the positivity of polynomials and related probabilistic and geometric ideas—see e.g. **[12]**—and that of Jones, **[13]**, Ocneanu, **[16]**, and others on the classification of subfactors of the hyperfinite factor of type II_1. Finally, one should certainly note the observation of Connes in **[3]** that a dimension group can be associated with the Penrose tiling.

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