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ETA-INVARIANTS AND VON NEUMANN ALGEBRAS

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1. Main theorem. Let M be a compact oriented Riemannian manifold of dimension $4k-1$. The operator $D = {*d} + d^*$ acting on the $2k-1$ forms on M is selfadjoint, and for any Hermitian vector bundle $E \to M$ with connection ∇, there is a selfadjoint operator $D \otimes \nabla$ acting on the smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E$. The data (D, E) defines a class $[D, E]$ in the odd analytic K-homology group $K_1^{an}(M)$. We develop in this work an equality between two methods of pairing $[D, E]$ with real-valued K^1-cohomology classes.

Let $\Gamma = \pi_1(M)$ and $A : \Gamma \to U_N$ be a representation, which determines a flat principal U_N-bundle $P_A \to M$ and an associated flat C^N-vector bundle $E_A \to M$. If there exists a bundle trivialization $\theta : P_A \cong M \times U_A$, then it is well known that the pair (A, θ) represents a class $\overline{A}$ in $K^1(M) \otimes R$, (§5, **[APS 2]**). The first pairing of $[D, E]$ and $\overline{A}$ uses the relative eta-invariant to define the flat bundle index, as in (§5, ibid). Let ∇^0 be the flat connection on $M \times U_N$ associated to the product structure, and let ∇^1 be the flat connection associated with the push-forward under θ of the flat connection on P_A. Define a smooth, one-parameter family of selfadjoint operators on smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E \otimes C^N$ by $D_t = t \cdot D \otimes \nabla \otimes \nabla^1 + (1-t) \cdot D \otimes \nabla \otimes \nabla^0$. The eta-invariant $\eta(D_t)$ is smooth as a function of t except for a finite number of bounded jump discontinuities, so there exists a well-defined continuous derivative function $\eta(D_t)'$ **[APS 1,2]**. Define

$$\langle [D, E], \overline{A} \rangle_\eta = \int_0^1 \eta(D_t)'\, dt. \tag{1}$$

The second pairing uses the von Neumann algebra $W^*(\Gamma)$ associated to the universal covering $\tilde{M}$ of M **[A 1]**. The lift to $\tilde{M}$, $\widetilde{D \otimes \nabla}$, of the operator $D \otimes \nabla$ is essentially selfadjoint, and we introduce the projection, P^+, from $H = L^2(\tilde{M}, \Lambda^{2k-1}(T^*M) \otimes E \otimes C^N)$ onto the positive space H^+ of $D \otimes \nabla$. The operator P^+ is Γ-invariant, so defines an element $[P^+]$ in $K_1(W^*(\Gamma))$. The data (A, θ) defines a map $u : \tilde{M} \to U_N$ by restricting the composition $p_2 \circ \theta : P_A \to M \times U_N \to U_N$ to the leaf of F_A on P_A through a basepoint, where F_A is the U_N-invariant foliation of P_A associated to the flat structure. We pair $[P^+]$ to $[u] \in K^1_{(\infty)}(M)$ by constructing an associated Toeplitz operator, which is Fredholm in the sense of Breuer, and has a von Neumann index which is the continuous dimension of the

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spectral flow between P^+ and the conjugate $u^{-1} \cdot P^+ \cdot u$. More precisely, multiplication by u on the coefficients C^N defines a bounded operator M_u on H, and the compression

$$T(u) = P^+ \circ M_u : H^+ \to H^+$$

is Γ-Fredholm and Γ-invariant, so as in **[A 1]** has a Γ-index. Define

$$\langle [D, E], \overline{A} \rangle_\Gamma = \mathrm{Ind}_\Gamma(T(u)). \tag{2}$$

The Γ-index for a fixed flat bundle P_A has an alternate description in terms of the type II von Neumann algebra $W^*(F_A)$ associated to the foliation F_A by Connes **[C 1]**, with trace derived from the Haar measure on U_N. The operator $T(u)$ is one of a U_N-parametrized family of leafwise Toeplitz operators along F_A, giving index data which is analogous to the hull-completion formulation of the index of almost periodic operators **[CDSS]**.

Our main result is

THEOREM.

$$\langle [D, E], \overline{A} \rangle_\eta = \langle [D, E], \overline{A} \rangle_\Gamma. \tag{3}$$

Moreover, both sides of (3) *are equal to a renormalized discrete spectral flow for the lift of* $D \otimes \nabla$ *to a nonelliptic, pure-point-spectrum* U_N*-invariant operator on* P_A.

The theorem is true also for any "geometric operator" on M which is obtained by coupling the Spinor Dirac operator on M to a coefficient vector bundle E, so that the theorem applies equally well, for example, to Spin manifolds of odd dimension.

The possibility of relating the eta-invariant for a family D_t to a von Neumann dimension as given by (3) above was suggested in (Remark 4, p. 89, **[APS 2]**).

2. Method of proof. There are three key points to the proof.

(2.1) The invariant $\int_0^1 \eta(D_t)'\,dt$ has a multiplicative property: For $E_2 \to M$ a flat vector bundle with fiber dimension q and flat connection ∇^2,

$$\int_0^1 \eta(D_t \otimes \nabla^2)'\,dt = q \cdot \int_0^1 \eta(D_t)'\,dt. \tag{4}$$

We take for E_2 the infinite-dimensional flat bundle associated to the composition of A with the left regular representation of U_N on $L^2(U_N)$. The Peter-Weyl Theorem decomposes this infinite bundle into an infinite direct sum of finite-dimensional bundles associated to the characters of irreducible representations of U_N. The equality (4) is interpreted in a renormalized sense by using the heat kernel, H_s, on U_N. We define a central eta-distribution on U_N by associating to a class function the weighted sum of the eta-invariants for the operators on M obtained by coupling $D \otimes \nabla$ to the finite-dimensional flat bundles associated to the characteristic subspaces of $L^2(U_N)$ in the Peter-Weyl decomposition. Then (4) becomes

$$\int_0^1 \eta(D_t; H_s)'\,dt = H_s(e) \cdot \int_0^1 \eta(D_t)'\,dt. \tag{5}$$

(2.2) The left-hand side of (5) can be written as the sum of two terms: a spectral flow summand, plus the difference of two values of the eta-distribution, one associated to $(D \otimes \nabla) \circ H_s$, and the other to $(D \otimes \nabla) \circ (u^{-1} H_s u)$ where now u is the operator on $L^2(U_N) \otimes C^N$ given by multiplication $uf(g) = g \cdot f(g)$ on the coefficients. Theorem 0.1 of **[CG]** implies that after renormalizing, the values of the eta-distribution on these two asymptotic functions agree, so that

$$\lim_{s \to 0} H_s(e)^{-1} \int_0^1 \eta(D_t; H_s)' \, dt = \mathrm{Ind}_{\mathrm{II}}([D, E], \overline{A}) \tag{6}$$

where $\mathrm{Ind}_{\mathrm{II}}([D, E], \overline{A})$ is the renormalized spectral flow of the eta-distribution for the family $\hat{D}_t$ of operators obtained by lifting $D_t \otimes \nabla$ to P_A via the leaves of F_A which cover M. This renormalized spectral flow is also the index, in a suitably renormalized sense, for a Toeplitz operator associated to the p.p.s. operator $\hat{D}_0$ and the multiplier M_u. We call this a renormalized transverse index, as it is based on the construction of the odd K_1-class for a transversally elliptic operator for the group U_N-action on P_A, in analogy with the even transversal index theory of **[A 2, S 1, 2]**.

(2.3) The third point of the proof is to use a Fubini principle, applied to the trace of kernels for operators on P_A, to show that the renormalized index $\mathrm{Ind}_{\mathrm{II}}([D, E], \overline{A})$ is equal to the index of the leafwise Toeplitz operator on F_A described in §1. The Weyl asymptotic theorem provides the final step, establishing that the renormalized trace converges to the foliation algebra trace formed from Haar measure on U_N.

3. Final remarks. The main theorem is part of the authors' study of analytic invariants associated to selfadjoint operators on a manifold which are regularized by a foliation on the manifold. This includes both leafwise and transversally elliptic operators for foliations, and we conclude with a discussion of some aspects of this program.

The details of the proof of the main theorem are given in **[DHK 3]**, where we also define for much more general classes of operators than $D = {*d} + d^*$, two types of cyclic cocycles, based on viewing the operator $\widetilde{D \otimes \nabla}$ as either leafwise for F_A, or transverse for the action of U_N. A topological index theorem can be derived for the first, while renormalization of the second expresses the relative eta-invariant for $D = {*d} + d^*$. A Fubini principle for transverse foliations establishes equality of the first cocycle with the renormalization of the second, and as a corollary yields the index theorem for flat vector bundles of **[APS 2]**. This procedure can be applied more generally to yield topological formulas for higher order regularized spectral invariants (cf. **[D]**).

The interpretation of the main theorem applied to $U_N = U_1$ via quasi-periodic functions was given in **[DHK 2]**, where the main theorem was compared to the results of **[CDSS]** and a proof was given via Fourier analysis on U_1.

The index theorem for leafwise selfadjoint elliptic operators can be interpreted via Toeplitz extensions of the foliation C^*-algebra by the algebra of functions on the manifold. In **[DHK 4]** we identify the analytic index class as an extension in the spirit of the Brown-Douglas-Fillmore theory.

The odd index theorem for coverings is discussed in **[H 1]**, where we give an alternate proof of the main theorem in terms of the eta-invariant for coverings based on **[CG]**. The spectral properties of leafwise elliptic operators is studied in **[H 2]**; it is expected that the spectral flow interpretation of the eta will extend to all leafwise eta-invariants.

A preliminary report on this work appeared in **[DHK 1]**, where the main theorem was announced with the additional hypothesis that the fundamental group of M be amenable. The outline of our program and the result for the amenable case was discussed in a plenary address at the conference *Operator Algebras and Geometry* at the Mathematical Sciences Research Institute, Berkeley, June 1985.

REFERENCES

[A 1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32–33** (1976), 43–72.

[A 2] ——, *Transversally elliptic operators and compact groups*, Lecture Notes in Math., vol. 401, Springer-Verlag, Berlin and New York, 1974.

[APS 1] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry*. I, Math. Proc. Cambridge Philos. Soc. **77** (1976), 43–69.

[APS 2] ——, *Spectral asymmetry and Riemannian geometry*. III, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.

[AS] M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*, Publ. Math. Inst. Hautes Études Sci. **37** (1969), 305–326.

[CG] J. Cheeger and M. Gromov, *On the characteristic numbers of complete manifolds of bounded curvature and finite volume*, Differential Geometry and Complex Analysis, dedicated to H. E. Rauch (I. Chavel and H. M. Farkas, eds.), Springer-Verlag, Berlin and New York, 1985.

[CD] L. Coburn and R. G. Douglas, *On C^*-algebras of operators on a half-space*. I: *Index theory*, Publ. Math. Inst. Hautes Études Sci. **40** (1971), 59–67.

[CDSS] L. Coburn, R. G. Douglas, R. G. Schaeffer, and I. M. Singer, *C^*-algebras of operators on a half-space*. II: *Index theory*, Publ. Math. Inst. Hautes Études Sci. **40** (1971), 69–79.

[C 1] A. Connes, *The von Neumann algebra of a foliation*, Lecture Notes in Physics, vol. 80, Springer-Verlag, Berlin and New York, 1978, pp. 145–151.

[C 2] ——, *Non-commutative differential geometry*, I.-*The Chern character in K-homology*, Publ. Math. Inst. Hautes Études Sci. **62** (1986), 257–309.

[D] R. G. Douglas, *Elliptic invariants for differential operators*, Proc. Weyl Symposium, Durham, N.C., 1987.

[DHK 1] R. G. Douglas, S. Hurder and J. Kaminker, *The eta-invariant, foliation algebras and cyclic cocycles*, Math. Sci. Res. Inst., Berkeley, preprint, 1985.

[DHK 2] ——, *Toeplitz operators and the eta-invariant: the case of S^1*, Contemporary Math., vol. 19, Amer. Math. Soc., Providence, R.I., 1988, pp. 11–41.

[DHK 3] ——, *Cyclic cocycles, renormalization and eta-invariants*, Univ. of Ill., Preprint, 1988.

[DHK 4] ——, *The longitudinal cocycle and the index of Toeplitz operators*, I.U.-P.U.@I. Preprint, 1988.

[H 1] S. Hurder, *Eta-invariants and the index theorem for coverings*, Univ. of Ill., Preprint, 1988.

[H 2] ——, *Analysis and geometry of foliations*, Research Monograph, Univ. of Colorado prepublication.

[S 1] I. M. Singer, *Some remarks on operator theory and index theory*, Lecture Notes in Math., vol. 575, Springer-Verlag, Berlin and New York, 1977, pp. 128–138.

[S 2] ——, *Recent applications of index theory for elliptic operators*, Proc. Sympos. Pure Math., vol. 23, Amer. Math. Soc., Providence, R.I., 1971, pp. 11–31.

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