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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Stability in model theory, by Daniel Lascar. Translated by J. E. Wallington. Pitman Monographs and Survey in Pure and Applied Mathematics, vol. 36, Longman Scientific and Technical, Essex, 1987, 193 pp., $77.95. ISBN 0-470-20386-2

Officially, model theory was created in 1950 by Alfred Tarski [**Ta**], who described it as a subject that lies on the "borderline between algebra and metamathematics." Many see it, nowadays, as generalized algebra, a point of view that is strongly emphasized in this book.

There is one obvious connection between algebra and the concept of model. The familiar classes of algebraic structures are usually described as the class of *models* of a given list of axioms. Such are, for instance, the class of groups, the class of fields, the class of algebraically closed fields. In these examples and in many more, the axioms can be stated as *first order sentences* in a suitable language. Let us explain what we mean by this. A logical language L comes equipped with a supply of operation and relation symbols of given arities (thus, we have a language for groups, another one for fields, a third one for ordered fields, etc.); one of them is always the equality symbol '='. The first order sentences of L are statements that use these symbols as well as variables and are constructed by means of logical connectives ("not", "and", "or", etc.) and quantifiers ("for all x", "there exists x", where x is a variable). The variables are required to be of the *first order*, i.e., ranging over individual elements rather than, say, over sets of elements (by the way, this is a serious limitation as, for example, one cannot describe in this way the notion of simple group). Model theory studies the class of models of an arbitrary set Γ of first order sentences. Notice that L and, hence, Γ are allowed to have not only countable but also uncountable cardinalities, a fact that increases the scope of model theory. The set T of L-sentences that are true in all models of Γ is called the (first order) theory axiomatized by Γ (a trivial exercise: Γ and T have the same models).

The previous paragraph contains a very brief description of the framework of model theory. It would be premature to state, on the basis of this tiny bit, that model theory is generalized algebra. There are marked differences between the two fields. Model theory strives for the general, while algebra seeks properties peculiar to specific classes. Algebra is preoccupied with notions related to structural properties of models such as dimension, for example, while model theory cannot limit itself to such notions, as for many first order theories there is no satisfactory way to define them. On the other hand, model theory considers seriously cardinalities of models, while algebra is not really concerned with these. As an example, Tarski proved in 1928 (based on earlier partial results of Löwenheim and Skolem) a theorem of stunning generality. It says that if a theory T has an infinite model then it has one in every cardinality that is greater than or equal to $|T|$, the cardinality of T. Algebraists never considered such problems. To speak in more general terms, let $I(\lambda, T)$ be the number of nonisomorphic models of T having cardinality λ. Thus, given a first order theory T, $I(\lambda, T)$ is a function of λ. Tarski's theorem and many other model theoretic results deal with properties of $I(\lambda, T)$, while algebraists find this function of little or no interest. But, surprise! Since the early sixties it was realized that certain assumptions about $I(\lambda, T)$ imply structural properties of the models of T, i.e., properties of the kind that interests algebraists. This development led to the creation of a rich mathematical theory, a significant part of which is expounded in this book. The subject has been stimulated by a few fortuitious questions, all related to $I(\lambda, T)$.

A theory T is called categorical in power λ if $I(\lambda, T) = 1$. The first question was: Is it true that if T is categorical in some $\lambda > |T|$ then it is categorical in every cardinality greater than $|T|$? It was asked for *countable* T by Łos in 1954, **[Lo]**, and answered in the positive by Morley in 1962, **[Mo]**. Morley's beautiful answer came somewhat as a surprise since Łos' conjecture was formulated on the basis of very few examples (the major one was the theory of algebraically closed fields of characteristic 0). Even less expected was the wide scope of the methods initiated in that work. In fact, Morley laid the cornerstone of a magnificent edifice. He showed that all countable theories categorical in some uncountable power have a certain property called by him "total transcendentality." Nowadays, we prefer the shorter name of "ω-stability." ω-stable theories are not necessarily categorical in some uncountable power but have good algebraic properties. One can define in their models quite satisfactory notions of independence of elements (generalizing algebraic independence) and of a model prime over a set (generalizing the notion of a structure generated by a set).

Morley left open the question whether Łos' conjecture is true for uncountable theories as well. Partial answers were obtained by Rowbottom (in an unpublished work in which he coined the term "stability"), Ressayre **[Re]** and, independently, Shelah **[Sh1]**. The full positive answer was established by the last mentioned author in 1971, **[Sh2]**. As emphasized also in the introduction of the book under review, stability theory as we know it today is the almost singlehanded creation of Saharon Shelah.

A few additional problems provided the impetus for further development. A most remarkable one, judging by its implications, was the following question asked by Morley: is $I(\lambda, T)$ a monotonically increasing function of λ, for $\lambda > |T|$? Let us remark that we may restrict this question to *complete* first order theories (a theory T in a logical language L is called complete if, for any L-sentence φ, T implies either φ or the negation of φ).

It took Shelah more than ten years to establish Morley's conjecture for countable theories (the uncountable case is still open). At a quite early stage, he made the bold conjecture that any complete theory T falls into one of the following two *disjoint* cases:

(a) ("nonstructure case") $I(\lambda, T) = 2^\lambda$ for all $\lambda > |T|$, i.e., T has the maximal number of models in every power greater than $|T|$, or: (b) ("structure case") the structure of all models of T can be described in a way that is detailed enough to allow us to count the number of models and conclude that Morley's conjecture holds.

There is no common ground, according to Shelah's conjecture, between having the maximal number of models and having a good-structure description. There is therefore, a *gap* between the theories of the first kind and those of the second; Shelah called it "the main gap."

It is instructive to compare the two conjectures. Morley's is a precisely formulated mathematical statement; Shelah's is not. The vagueness of the latter turned out to be an advantage because it gave the fertile imagination of Shelah the latitude to crystallize the precise content of the "structure case" gradually, as the work progressed. As it turned out, every model could be characterized by a set of cardinal invariants, a sort of generalized dimensions.

According to Shelah's plan, the complete theories should be classified, i.e., divided into finitely many classes and each class of theories shown to belong either to the nonstructure or to the structure case. Whenever the latter occurs, the structure description should be elaborated until it allows us to "pass the test" of establishing Morley's conjecture.

This plan has been carried out in stages that have been described, over the years, in various publications. Thus, the landmark book **[Sh3]** contained only a partial solution. Stronger results were established in **[Sh4, Sh5]** and the full countable case of Morley's conjecture awaits its publication in the second edition of **[Sh3]**.

The book under review is concerned with the structure case only. This is the context in which model theory can be best viewed as generalized algebra, because the main preoccupation is with structure description. The nonstructure theorems are stated without proofs. Even considering this limitation of scope, it is amazing how much material has been amassed in this booklet. The determined reader with a background in general model theory can start reading the book and if he labours his way through 183 small typewritten pages, he will find himself familiar with most notions and with the spirit of a field that is currently active. The main "structure case" results of **[Sh3, Sh4, Sh5]** are presented. Inevitably, some "hot" topics had to be left out; these include recent work on the structure of countable models as well as the subject of stable groups. Still, the principal tools are present.

The careful exposition is heavily influenced by the author's own contributions. This is felt especially in Chapter 3–4 and 7–8; in particular, Chapters 2 and 3 present the elegant Lascar-Poizat definition of forking. One should also mention the author's preference for using the language of topology.

The general reader who is interested in more details at a nontechnical level, should read the nice introduction of the book. Here are two comments meant to bridge between this review and Lascar's account.

First of all, Morley's conjecture is not mentioned at all in the book. In contrast, Shelah's is given the deserved prominence. This approach, which is quite customary among the researchers in the field, is justified by the fact that the latter conjecture was the immediate motivation for the whole theory.

The second comment concerns nomenclature. Shelah introduced the term "classification theory" to indicate that first order theories are divided into classes. Lascar and other authors give "classification" a different meaning by saying that the "structure case" theories are those whose models can be "classified."

There is one omission of a technical nature that, in my opinion should have been avoided. Shelah devised a construction that yields, given any theory T, another theory T^{eq} that is closely related and more pleasant to work with. This construct is absent from the book; thus, the author shows us that it is not necessary, at least for the results that he describes. On the other hand, T^{eq} occurs frequently in research papers (see **[La]**, for example).

The worthy task of presenting Shelah's deep theory in an expository form is a difficult one and this small book is a welcome addition to the voluminous and more ambitious **[Sh3** and **Ba]**.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Hilbert modular surfaces, by Gerard van der Geer. Ergebnisse der Mathematik, Band 16, Springer-Verlag, Berlin, Heidelberg, New York, 1988, ix + 291 pp., $69.50. ISBN 3-540-17601-2

The theory of Hilbert modular surfaces is a generalisation of the classical theory of automorphic forms, and in many ways it is one of the easiest generalisations. It was started by D. Hilbert at the end of the last century, with the motivation to enhance the theory of analytic functions of several complex variables. He inspired O. Blumenthal to take up the subject, and sometimes his name is added to that of Hilbert. However, neither man got very far, and only after the general theory of complex manifolds had advanced sufficiently could progress be made in this special case. In modern times the subject has been revived by M. Rapoport and F. Hirzebruch. The theory of Hilbert modular surfaces mixes the theory of automorphic forms, arithmetic algebraic geometry (especially Shimura-varieties) and the theory of classical complex algebraic surfaces. It thus seems appropriate to give short overviews of recent developments in these subjects, and after that we try to explain how they specialize to the case of Hilbert modular surfaces. Needless to say, I tend to oversimplify the situation; for details one should consult the literature.

The theory of automorphic forms started with the classical elliptic modular functions (for a modern account see [**La**]), and has developed into a theory about reductive groups. Let us try to explain how this happened: Classically one considers the upper halfplane $\mathbf{H}$ of complex numbers with positive imaginary part, on which the group $\mathrm{SL}(2, \mathbf{R})/\{\pm 1\}$ acts by the usual $(az + b)/(cz + d)$-rule. One further chooses a subgroup Γ of finite index in $\mathrm{SL}(2, \mathbf{Z})$, and considers holomorphic functions $f(z)$ which transform under Γ according to a certain factor of automorphy, and which are holomorphic at infinity (this amounts to a certain growth-condition).