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Spectral theory and differential operators, by D. E. Edmunds and W. D. Evans. Oxford University Press, Oxford, 1987, xvi + 574 pp., \$115.00. ISBN 0-19-853542-2

A linear operator is a complicated mathematical object, but many of its properties are captured in its spectrum, which is simply a set of complex numbers. For a selfadjoint linear operator, the spectrum consists of real numbers and has a quantum mechanical interpretation. If the operator represents the energy of an atomic or molecular system, the spectrum represents possible energy levels of the system. Furthermore the differences between these energy levels determine the permitted frequencies of emitted light. The rich phenomena of color are determined by the spectra of linear operators.

One striking feature of such an energy spectrum is that it usually consists of two parts. For the hydrogen atom the first part is a sequence of negative energy levels proportional to $-1/n^2$, where $n = 1, 2, 3, \dots$, accumulating at zero. These correspond to the various bound states of the electron. The remaining part consists of all energies zero and above. These correspond to states of electrons that are free to escape.

In the general setting when H is a linear operator acting in a Banach space, the *spectrum of H* is defined as the set of complex numbers E such that $(H - E)^{-1}$ is not a bounded operator. (A *bounded operator* is an operator defined on the Banach space that maps every bounded set into a bounded set.) In particular, if E is an eigenvalue of H , then the inverse of $H - E$ does not exist, and so E belongs to the spectrum. However a number may be in the spectrum for other reasons; for instance even when $H - E$ is one-to-one it is possible that its range is not the entire Banach space.

One of the most elementary ways of dividing the spectrum into two parts is by introducing a subset called the *essential spectrum*. In the Banach space setting there are at least five inequivalent definitions of essential spectrum. Fortunately, in the case of a selfadjoint operator acting in a Hilbert space they are all equivalent. In this case the essential spectrum may be characterized as the part of the spectrum that does not consist of isolated eigenvalues of finite multiplicity. In the hydrogen atom example the essential spectrum is the interval from zero to infinity.

The reason for introducing the concept of essential spectrum is that the essential spectrum of a bounded operator is invariant under compact perturbations. (An operator is *compact* if it maps every bounded set into a set with compact closure.) This is often used in the following somewhat more general setting. Assume that

$$H = H_0 + V$$

and that V is *relatively compact* in the sense that for some fixed E not in the spectrum of H_0 the operator $V(H_0 - E)^{-1}$ is compact. If both $(H - E)^{-1}$

and $(H_0 - E)^{-1}$ are bounded, then there is an algebraic identity

$$(H - E)^{-1} = (H_0 - E)^{-1} - (H - E)^{-1}V(H_0 - E)^{-1}$$

relating the two bounded operators. Since the compact operators form an ideal in the ring of bounded operators, the second term on the right is compact. Therefore the essential spectrum of $(H - E)^{-1}$ is the same as the essential spectrum of $(H_0 - E)^{-1}$, and it follows that the essential spectrum of H is the same as the essential spectrum of H_0 . In summary, the essential spectrum is preserved under relatively compact perturbations.

These ideas are used in quantum mechanics when the operators H_0 and V represent kinetic energy and potential energy. These operators are defined on dense linear subspaces of the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable complex functions on \mathbb{R}^n and have values in this same Hilbert space. Let x represent the position coordinates on \mathbb{R}^n . The kinetic energy operator corresponding to mass $m > 0$ is

$$H_0 = -\frac{\hbar^2}{2m}\nabla_x^2.$$

Here $\hbar > 0$ is another constant, so H_0 is just a negative multiple of the Laplace operator. The constant \hbar has the units of angular momentum, so H_0 has the units of energy, as it should. Now let p represent momentum coordinates dual to the position coordinates, and write px for the scalar product with x , so that px/\hbar is dimensionless. Since

$$-\frac{\hbar^2}{2m}\nabla_x^2 \exp\left(\frac{ipx}{\hbar}\right) = \frac{p^2}{2m} \exp\left(\frac{ipx}{\hbar}\right),$$

the Fourier transform may be used to show that the operator H_0 is isomorphic to pointwise multiplication by

$$h_0(p) = \frac{p^2}{2m}.$$

Here p^2 is the square of the length of the momentum vector p , so this is the usual classical mechanical expression for kinetic energy. Since the range of this function is the interval from zero to infinity, the spectrum of H_0 consists of essential spectrum in this interval.

The quantum mechanical potential energy operator is pointwise multiplication by a real measurable function $V(x)$ on \mathbb{R}^n . The corresponding Schrödinger operator is the total energy operator

$$H = -\frac{\hbar^2}{2m}\nabla_x^2 + V(x).$$

It is remarkable that while the first term is isomorphic to multiplication by the function $h_0(p)$ and the second term is multiplication by the function $V(x)$, there is no isomorphism that turns them both into functions. This is the notorious complementarity of quantum mechanics.

This structure does have advantages; it explains the discreteness of spectra. If the function $V(x)$ is not excessively singular and approaches zero as $|x|$ tends to infinity, then the operator of multiplication by the function $V(x)$ is relatively compact. It follows that the essential spectrum of

the Schrödinger operator is also the interval from zero to infinity, and the spectrum below zero is discrete. The hydrogen atom is a good example. Its potential energy $V(x)$ is proportional to $-1/|x|$, the potential energy between two charges of opposite sign. For the usual case of space of dimension $n = 3$ the local singularity of $V(x)$ is not too severe, and the decrease at infinity provides the relative compactness.

Having isolated the essential spectrum and its complement, the next task is to examine each part. The complement consists of isolated eigenvalues with finite multiplicity and should be the easiest to study. In the quantum mechanical setting a common task is to find how many energy levels there are less than a given energy value E . Since the early days of quantum mechanics there was a notion that this should be approximately the corresponding classical phase space volume, measured in appropriate units. In other words, interpret H not as an operator but as a function

$$H = \frac{p^2}{2m} + V(x)$$

on classical phase space \mathbb{R}^{2n} with coordinates p and x , representing momentum and position. Evaluate the integral over the subset of phase space where $H < E$ with respect to the measure $d^n p d^n x / (2\pi\hbar)^n$. The condition for a point to be in the subset may also be written as $|p| < r(x)$, where

$$r(x) = \sqrt{2m[E - V(x)]_+}$$

and $[E - V(x)]_+$ is $E - V(x)$ when this is positive, zero otherwise. For fixed x this is a ball in \mathbb{R}^n . Therefore when we evaluate the momentum integral we obtain

$$\int \int_{H < E} \frac{d^n p d^n x}{(2\pi\hbar)^n} = \omega_n \int r(x)^n \frac{d^n x}{(2\pi\hbar)^n},$$

where ω_n is the volume of the unit ball in \mathbb{R}^n . One of the main results of the book under review is an estimate that shows in what sense this is a good approximation to the number of energy levels below E .

The book of Edmunds and Evans begins with four chapters on abstract operator theory. The authors treat compactness and the essential spectrum in some detail. For example they report a result that even though there are five different definitions of essential spectrum, the radius of the essential spectrum, however defined, is always the same. There is also material on unbounded linear operators and on sesquilinear forms in Hilbert space.

These abstract developments are followed by a chapter on Sobolev spaces and Sobolev inequalities. In some situations these spaces have a quantum mechanical interpretation. For instance, if a complex function ψ on \mathbb{R}^n is sufficiently smooth and vanishes at infinity in the appropriate sense, then integration by parts shows that the expected kinetic energy is

$$\langle \psi, H_0 \psi \rangle = \int \bar{\psi} \left(-\frac{\hbar^2}{2m} \nabla_x^2 \right) \psi d^n x = \frac{\hbar^2}{2m} \int |\nabla_x \psi|^2 d^n x.$$

A function ψ is in the Sobolev space of functions with one derivative in $L^2(\mathbb{R}^n)$ if the right hand side of this expression is finite. The corresponding

Sobolev inequality says that when $n > 2$ such a ψ must be in $L^p(\mathbb{R}^n)$, where $2/p + 2/n = 1$. By Hölder's inequality this implies that if a potential energy function V is singular but in $L^{n/2}(\mathbb{R}^n)$, then the expected potential energy $\langle \psi, V\psi \rangle$ can be controlled in terms of the expected kinetic energy. Of course the mathematical significance of Sobolev spaces goes far beyond quantum mechanics; in the words of the authors, they "provide a natural framework for much of the modern theory of partial differential equations."

The rest of the book is on second order elliptic linear partial differential operators. There are two chapters on boundary value problems in the Hilbert space setting. The remaining five chapters are devoted to eigenvalue problems. There is a presentation of Molcanov's necessary and sufficient condition for a Schrödinger operator to have a wholly discrete spectrum. (It is clear that $V(x)$ must approach infinity as $|x|$ approaches infinity in some sense, but the precise statement involves the notion of capacity.) This is followed by results on the location of the essential spectrum. The culmination of this development is a general bound on the difference between the number of eigenvalues below E and the corresponding volume in classical phase space. This is obtained by localization to cubes. As corollaries the authors obtain results on the asymptotics of the number of eigenvalues below E as this number approaches infinity as well as the Cwickel-Lieb-Rosenbljum upper bound for the number of eigenvalues. The book concludes with a discussion of singular values for Schrödinger operators with complex potentials.

While a major application of this theory is to energy levels of quantum mechanical systems, the book attempts to put the results in a more general context. Thus most of the abstract theory is in the Banach space setting, and even when the operators are restricted to operate in Hilbert space, they tend not to be selfadjoint.

There is a price to this generality. The division of the spectrum into essential spectrum and its complement is reasonable when the complement is of principal interest, as in the problem of counting eigenvalues. However there are many remaining questions about the essential spectrum. For instance, it is reasonable to ask whether this part of the spectrum is continuous spectrum (corresponding to scattering states) or dense point spectrum (corresponding to a sea of bound states). Unfortunately, it is difficult to approach these problems without specializing to selfadjoint operators acting in a Hilbert space. Of course the price of generality may be worth paying, since there are significant uses for linear partial differential operators that are not selfadjoint. The book, however, has little to say about applications other than quantum mechanics, and even the quantum mechanics is somewhat of a side issue. Its subject is partial differential operators; it will be a substantial treat for lovers of this area of analysis.

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