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An introduction to the theory of the Riemann zeta-function, by S. J. Patterson. Cambridge Studies in Advanced Mathematics, vol. 14, Cambridge University Press, Cambridge and New York, 1988, xiii + 156 pp., $34.50. ISBN 0-521-33535-3

Introduction to analytic number theory, by A. G. Postnikov, with an appendix by P. D. T. A. Elliott. Translated by G. A. Kandall. Translations of Mathematical Monographs, vol. 68, American Mathematical Society, Providence, R.I., 1988, vi + 320 pp., $114.00. ISBN 0-8218-4521-7

Problems concerning the distribution of prime numbers go back to antiquity. Solutions to them seem to be elusive. At the age of 17, Gauss conjectured that the number of primes $\leq x$, denoted $\pi(x)$, should be asymptotic to

$$\operatorname{li} x = \int_2^x \frac{dt}{\log t}.$$

In a classical ten page paper of 1858, Riemann introduced the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1)$$

as a function of a complex variable and showed how the analytic properties of $\zeta(s)$ should prove the asymptotic law conjectured by Gauss. This was the beginning of the persistent theme of L-series in number theory, the interplay of analysis and arithmetic. The fundamental relation connecting the ζ-function with prime numbers is the "Euler product"

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over primes p. This relation is equivalent to the unique factorization of natural numbers. In his seminal paper, Riemann derived an analytic continuation of $\zeta(s)$ as a meromorphic function of s with only one singularity, a simple pole at $s = 1$. By using the modular transformation of the theta function, he derived the functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1/s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Without giving any details, he wrote down an infinite product expansion for $\zeta(s)$,

$$s(s-1)\Gamma(s/2)\zeta(s) = e^{bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)e^{s/\rho}$$

where ρ runs through the zeroes of $\zeta(s)$ satisfying $0 \leq \Re(\rho) \leq 1$. In the same spirit, he wrote down an asymptotic formula for the number of such zeroes s satisfying $0 \leq \Im(s) \leq T$. The most astounding idea of the paper was to connect this product representation together with the Euler product to derive an explicit formula for the number of primes up to x: let

$$f(x) = \operatorname{li} x - \sum_{\rho, \Im(\rho)>0} \{\operatorname{li} x^{\rho} + \operatorname{li} x^{1-\rho}\} + \int_x^{\infty} \frac{dt}{(t^2-1)\log t} - \log 2.$$

Then, $f(x)$ enumerates the number of integers $n \leq x$ which are prime or a power of a single prime. Thus,

$$\pi(x) = \sum_{n=1}^{\infty} \mu(n) f(x^{1/n}),$$

where μ denotes the Möbius function. From this, one can derive the asymptotic law conjectured by Gauss, and the main contribution comes from the simple pole at $s = 1$. In the later years of the nineteenth century, a galaxy of mathematicians such as Mertens, von Mangoldt, Hadamard, and de la Vallée-Poussin developed these ideas and their work culminated in the proof of conjectured law of Gauss, the celebrated prime number theorem.

At the conclusion of the same paper, Riemann makes his famous hypothesis: all of the zeroes ρ of $\zeta(s)$ such that $0 \leq \Re(\rho) \leq 1$ must have $\Re(\rho) = 1/2$. It seems an understatement to say that this single paper has guided, inspired and shaped much of modern mathematics, first in giving a thrust to the development of complex analysis, second in developing the analogues over number fields and function fields over finite fields, and third in encoding algebraic information in the zeta function thereby forging a synthesis of algebra, arithmetic and analysis. A sweeping glance at the development of mathematics after Riemann conjures up a stream of ideas of Hecke, Artin, Weil, Tate, Selberg, Grothendieck, Deligne and Langlands, all of whose powerful work emanates from the concept of a zeta function.

Patterson's book seeks to understand the zeta function with these final developments kept in mind in the background. It is however intended as an introductory text. After a brief historical introduction, Patterson uses the Poisson summation formula to derive a functional equation for the "zeta function"

$$M(f, s) = \int_{\mathbf{R}} f(x)|x|^{s-1}\, dx$$

where f is a measurable function of suitable growth. He derives the formula

$$M(f, s) = \gamma(s) M(\hat{f}, 1-s), \qquad \gamma(s) = \pi^{1/2-s}\Gamma(s/2)/\Gamma((1-s)/2),$$

where $\hat{f}$ is the Fourier transform of f. This is reminiscent of Tate's thesis. In fact, the celebrated thesis brings out this Fourier duality behind all functional equations of abelian L-series.

The subsequent chapters are devoted to a rigorous derivation of the statements of Riemann, ending with the proof of the explicit formula and a proof of the prime number theorem.

The fifth chapter is certainly the most interesting. It is a discussion of the Riemann and Lindelöf hypotheses. The author says there are several reasons for believing the Riemann hypothesis. The first 1.5 billion zeroes lie on the critical line. The work of Selberg and Levinson shows that a positive proportion of the zeroes lie on the critical line. The zeta functions of function fields over finite fields as initiated by E. Artin satisfy the analogue of the Riemann hypothesis. Thus, Chapter five is devoted to proving the function field analogue using an approach outlined by Weil. In a paper written in 1958, Weil derived the explicit formula in a more general context and defined a hermitian form R. The Riemann hypothesis is true if and only if R is positive semidefinite. The most striking feature of this formulation is that when applied to nonabelian L-series the positive semidefiniteness of R implies in addition the famous Artin conjecture on the holomorphy of Artin L-series. The novel feature of this book is that it gives the proof in the function field case based on these ideas in the hope that it may shed some light in the classical case. Certainly §§5.17, 5.18, 5.19 are most stimulating even to the expert. In this context, it is worthwhile to read Iwasawa's article [3] which discusses the motivation for the Iwasawa theory, much of which was inspired by these analogies.

The last chapter deals with the approximate functional equation which is a powerful tool in obtaining finer results in the theory of the zeta function.

All in all, this is a beautiful introduction to analytic number theory and is highly recommended for a first course in number theory at the graduate level. There are numerous exercises at the end of every chapter and no prerequisites beyond basic analysis. It is not a treatise on the analytic theory of the zeta function like Titchmarsh [4] or Ivić [2], or a historical survey, like the book by Edwards [1]. It emphasizes the role of the zeta function against the background of developments of both number theory and algebraic geometry.

In contrast, Postnikov's book *Introduction to analytic number theory* serves as an introduction only to one who wants to pursue additive methods in analytic number theory. In addition to fundamentals of analysis and number theory, it requires a background in probability theory. It is an axiomatic treatment of the theory of arithmetic functions. He considers the following setting.

Let G be a multiplicatively written semigroup with a countable system P of generators $\omega_1, \omega_2, \dots$, and suppose given a homomorphism N of G into the positive real numbers. Moreover, suppose that there are only finitely many elements $\alpha \in G$ such that $N(\alpha) \le x$. This situation can be viewed as an abstract version of the natural numbers with P being viewed as an analogue of the set of prime numbers. Define, in this analogy,

$$\pi_G(x) = \sum_{\substack{\omega\in P\\ N(\omega)\le x}} 1, \qquad \nu_G(x) = \sum_{\substack{\alpha\in G\\ N(\alpha)\le x}} 1.$$

Then an additive theorem of Bredikhin states that if

$$\pi_G(x) = \tau\frac{x}{\log x} + O\left(\frac{x}{\log^{1+\varepsilon} x}\right),$$

then

$$\nu_G(x) = C_G x \log^{\tau-1} x + O\left(\frac{x \log^{\tau-1} x}{(\log\log x)^{\varepsilon_1}}\right)$$

where $\varepsilon_1 = \min(1, \varepsilon)$, and C_G is a constant. In a similar vein, the author considers the inverse problem. Again Bredikhin showed that if $\nu_G(x) = C_G x^\theta + O(x^{\theta_1})$, where $0 \leq \theta_1 < \theta$, then

$$\pi_G(x) \sim \frac{1}{\theta}\frac{x^\theta}{\log x}.$$

Such theorems form the theme of Chapters one and two.

In Chapters three and four, the author considers the problem of estimating the averages of general arithmetic functions. The prototype of such a theorem is the classical theorem of Wirsing. Suppose $f(n)$ is a multiplicative function such that $f(n) \geq 0$, $f(p^r) \leq C_1 C_2^r$, $C_2 < 2$, and $\sum_{p \leq x} f(p) \log p \sim \tau x$. Then as $x \to \infty$,

$$\sum_{n \leq x} f(n) \sim \frac{e^{-\gamma\tau} x}{\Gamma(\tau) \log x} \prod_{p \leq x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right).$$

Such theorems are potentially useful in situations when we do not know the analytic properties of the corresponding Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Much of the book is technical and is suitable only to the expert in additive number theory. It should be remembered that this monograph was originally written in Russian in 1971. The present book under review is a recent AMS translation which has an updated appendix by P. D. T. A. Elliott.

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M. RAM MURTY
MCGILL UNIVERSITY