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Ergodic theory and differentiable dynamics, by Ricardo Mañé, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 8, Springer-Verlag, Berlin, 1987, xi + 317 pp., $82.00. ISBN 0-387-15278-4

In 1931 G. D. Birkhoff published the proof of one of the most profound theorems of this century **[B]**. This theorem, which has come to be known as the Birkhoff ergodic theorem, is remarkable in several ways. It is the only recent instance which comes to mind of a single theorem giving rise to a whole new branch of mathematics. Moreover it is one of those rare theorems whose content and significance can largely be understood by nonmathematicians.

The motivation for the ergodic theorem came from the work of Boltzmann and Gibbs on statistical mechanics. The mathematical question arising from their work was under what conditions the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x))$$

exists and is independent of $x \in X$, where $f : X \to R$ is a real valued function on a space X and $T : X \to X$ is a transformation. This limit is the average value of the function f along the forward orbit of the transformation T.

Birkhoff's theorem concerns the case when (X, μ) is a finite measure space, f is measurable, and T is a measurable transformation for which the equation $A = T^{-1}(A)$ is never satisfied unless A has measure 0 or full measure. Such transformations are called *ergodic*. The theorem is often paraphrased by saying that for ergodic transformations the time average equals the space average. In other words if we consider the transformation T as a dynamic which occurs every unit of time, then for almost all starting points $x \in X$ the average value of the function f on the orbit of x as it evolves through time exists and is equal to $\int_X f d\mu$, the average value of the function f on the space X. Intuitively, if we consider the case when f is 1 on a measurable set A and 0 elsewhere, then this says that a typical particle x in an ergodic system will spend a proportion of its time in A equal to the proportion of the total volume in A. Except for the technical concept of measure 0 implicit in the conclusion about almost all x, this is easily explainable to a nonmathematician.

More precisely, Birkhoff's theorem asserts that for any finite measure space (X, μ), measurable function f and measure preserving transformation T, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x))$$

exists for almost all x. Moreover, if T is ergodic then the value of this limit is the same for almost all x and in fact

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x)) = \int_X f d\mu.$$

The field of ergodic theory can fairly be said to owe its existence to this theorem. During the last twenty-five years special attention has been paid to smooth transformations $T : X \to X$ with X a manifold and to results displaying an interaction between ergodic properties of T and its smoothness. This body of work is the focus of Mañé's book.

When X is a compact metric space and $f : X \to X$ is continuous there is always at least one ergodic f-invariant measure μ defined on X. (This measure and all the others we discuss are Borel measures, that is, they are defined for sets in the σ-algebra generated by open and closed sets.) However, if X is a smooth manifold it may come equipped with a natural smooth measure which may not be invariant under f and may have nothing to do with an ergodic μ. In fact the smoothness structure on X at least determines a class of measures—those which come from smooth volume forms on X. The difficulty is that a set which has full measure with respect to an ergodic invariant μ may have measure 0 with respect to any smooth measure. Hence a result about almost all points with respect to μ may not be very interesting. We would like to say something about the time average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x))$$

for all x in a set of full measure with respect to a smooth measure. (All smooth measures have the same sets of measure 0.)

This is possible if an ergodic invariant measure μ can be found which has the same sets of measure 0 as smooth measures. Mañé works through several interesting examples where such measures can be found. For example, if $f : [0, 1] \to [0, 1]$ is the Gauss transformation, given by

$$f(x) = \frac{1}{x} - [x],$$

then an ergodic invariant measure is defined by

$$\mu(A) = \frac{1}{log(2)} \int_A \frac{dx}{1+x}.$$

It turns out that this example is intimately related to the continued fraction expansion of real numbers and Mañé gives us some insight into how ergodic theory can have applications in number theory.

The technique used to obtain the measure μ is susceptible to generalization and this generalization is carried out culminating in a proof of the existence of a measure μ with the desired properties for quite a large class of smooth transformations. Throughout the exposition is good and the role played by differentiability of the transformation is always stressed.

However, for most smooth transformations there is no invariant measure which has the same sets of measure 0 as smooth measures. In this case it may still be possible to find an ergodic invariant measure μ such that the conclusion of the ergodic theorem,

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x)),$$

holds for all $x \in X$ except a set of smooth measure 0. The search for such measures is a topic of considerable current research interest, especially for so-called "strange attractors." Mañé takes us through the construction of such a measure in a prototypical case (when T is an Anosov diffeomorphism) in a way that builds on the same tools and techniques used for the earlier examples. Moreover in this case the measure μ satisfies

$$\mu(A) = \lim_{n\to\infty} \lambda(T^{-n}(A))$$

for any smooth measure λ so that it is (at least in principle) calculable from a smooth measure.

The main tool used in most of these constructions is the existence of partitions of X which satisfy certain nice properties with respect to the smooth transformation T. The final chapter of this book is devoted to a numerical invariant of a measure preserving transformation which is defined in terms of partitions. This invariant $h_\mu(T)$ was introduced into ergodic theory by Kolmogorov and is called the entropy of T. It has proven to be one of the most important invariants in ergodic theory and it plays an especially important role in the ergodic theory of smooth dynamical systems. Surprisingly the entropy, which is a purely measure theoretic construct, has an intimate connection with smooth aspects of the dynamics of T and even the algebraic topology of T.

I like this book very much. It presents the right material in a coherent framework with just the right balance of general theory and concrete examples. The topics covered are well chosen and there is a wealth of material to be found here. It contains much more than can be described in this brief outline, but the topics above form the trunk from which branches emanate.

The main fault I found with this book is that it contains an unusually large number of typographical errors. All those I found could easily be recognized as typographical errors and I had no difficulty in knowing what was intended. Nevertheless, it can be extremely disconcerting and even confusing to encounter typographical errors in a difficult mathematical proof.

It must be said that some of the topics covered by Mañé (the ergodic theory of Anosov diffeomorphisms for example) are technical and difficult.

This is probably a difficulty with any book presenting a young and active field, however. The author in his introduction acknowledges that some proofs are "arid and demanding" and encourages the reader to concentrate on their complement in the text. This is good advice, and I found it quite feasible to do so and still learn a great deal.

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Lectures on counterexamples in several complex variables, by John Erik Fornaess and Berit Stensønes. Mathematical Notes 33, Princeton University Press, Princeton, N. J., 1987, 247 pp., $22.50. ISBN 0-691-08456-4

This book is about examples in several complex variables, called for some strange reason counterexamples. This is a very nice and useful book, which gathers examples to be collected in many different places. It starts (the first 65 pp.) with an elegant survey, with proofs, of the most basic results about holomorphy, subharmonicity, and pseudoconvexity (including some material not to be found in the presently available textbooks). However, one sees once more the tremendous and amazing resistance to defining subharmonic functions by the fact that their Laplacian, in the sense of distributions, is a positive measure.

Although strict pseudoconvexity of a domain in C^n is a simple notion (a domain is strictly pseudoconvex if and only if, locally, it can be made strictly convex, under a holomorphic change of variables), the notion of (weak) pseudoconvexity is more subtle. The first basic example in this area is the Kohn-Nirenberg example which shows, in particular, that pseudoconvexity is not "locally equivalent" to convexity (after holomorphic change of variables). In fact, one can even start thinking about the crucial relation between convexity and subharmonicity in one complex variable. It is easy to see that a twice continuously differentiable strictly subharmonic function (Laplacian strictly positive) can be made strictly convex, in the neighborhood of any noncritical point, by a local holomorphic change of variables. The same fails to be true for subharmonic functions.

The world of weak pseudoconvexity had to be explored: exhaustion functions, neighborhoods .... Several examples by Diederich and Fornaess (including the famous worm domain) constituted a major achievement in this area. It is one of the main topics in the book.