BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Fundamentals of stability theory, by John T. Baldwin. Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York, 1988, xiii + 447 pp., $60.00. ISBN 3-540-15298-9

1. The theory of classification of structures. I first met Saharon Shelah at the ICM in 1974. When I mentioned a result on universal locally finite groups which had been obtained recently by model theoretic methods, his reaction startled me: "Oh! I think I can prove a much better theorem, but tell me—what *is* a universal locally finite group?" At that time Shelah had a clear view of a very general theory of the construction and classification of algebraic systems of quite general types, based on remarkably superficial algebraic information, and less superficial model theoretic considerations. In the case of universal locally finite groups he applied this theory to justify his claim on the spot, after hearing the definition and a sketch of the earlier argument.

The problem which Shelah's "classification theory" (also referred to as "stability theory") is intended to solve is the following. A class K of algebraic structures is specified, for example the class of universal locally finite groups (whatever that may mean). If the isomorphism types of the structures in K can be classified in terms of intelligible numerical invariants, the theory should enable us, in principle, to find this classification. If such a classification is not possible, the theory should provide convincing evidence of this fact by enabling us to construct many structures in the class. The term "many" may mean various things, for example that a completely arbitrary structure can be set-theoretically encoded into the isomorphism type of a structure in the specified class.

Shelah calls the critical dichotomy in classifcation theory a "structure/nonstructure" theorem. The basic result of classification theory is that for certain reasonable classes of algebraic structures, the isomorphism types of structures in the class are either classifiable or quite wild; there is no middle ground. The paradigm for the "good" case is the class of algebraically closed fields, and analogs of transcendence degree play a role in *all* good cases, whereas a paradigm for the "bad" case is the family of linear orderings; the set-theoretic methods used originally to construct many complicated linear orderings have been extended by Shelah to all cases in which there is no structure theory.

For classification theory to fulfill its intended purpose, it should apply to quite general classes K of algebraic structures. Indeed, to have a rigorous formulation of classification theory at all, it is necessary at the outset to specify the context, that is to define the admissible families K to which the theory will apply. The fundamental theorem of classification theory will then be a rigorous structure/nonstructure theorem stating that every class K admitted by the specified context must fall on one side or the other of the stated dichotomy.

In its first incarnation, the context for classification theory was a (modified) first order context: we consider classes K consisting of all the (somewhat saturated) models of an axiom system that can be formulated in a classical, finitary, first order language. This condition may be excessively restrictive—for example the notion "locally finite" cannot quite be formulated in such a language, though the methods of the theory apply—but the book under consideration (like all books on the subject, so far) sticks to the first order context, and for good reason: the investigation of alternate contexts is still under way; the first order context still presents many open problems; and the ideas that emerge in the first order context dominate the subject. The paradigmatic form of classification theory was developed in **[11]**; and **[14]** describes a variant that would easily cover the case of universal locally finite groups, as well as more exotic classes of structures. It is a remarkable result of this theory that for the sort of classes under consideration, whenever an attempt at classification succeeds completely, the classification is necessarily in terms of invariants much like the ones familiar from standard examples.

There are alternate contexts for classification theory, and these are of two kinds. One approach is to look for more *general* contexts which capture other examples of concrete interest. From the point of view of model theory this is the main line, and it has been vigorously pursued by Shelah and his epigones. Naturally, this line leads to a more complicated version of the theory.

One may also look at more *concrete* contexts, arising for example in algebra, which are not necessarily special cases of the abstract context. For example the class K_R of all *pure-injective* modules over a fixed ring R, preferably broken up into the subclasses corresponding to first order invariants, provides a slightly more concrete version of the "non-multidimensional stable ε-saturated" case of the general theory, worked out vigorously by Ziegler, which at the same time connects to the study of finitely generated modules over artin algebras **[10]**. In a quite different vein, Lachlan has a theory of *finite* homogeneous structures **[5]** which is a "finitization" of classification theory according to the principle that "infinite" can be taken to mean "large enough." This version of classification theory lies entirely on the good side of the structure/nonstructure dichotomy, and produces a classification of each class admitted by the context into a finite number of infinite families, plus a finite number of examples. As it happens, though the guiding ideas of this theory are closely modelled on ideas which first arise naturally in the context of the model theory of infinite structures, to make them work for finite structures requires an additional ingredient: the classification of the finite simple groups.

Classification theory has meant a radical departure for model theory, and as such has spawned an extensive literature, much of it for internal consumption. There are also a number of articles aimed squarely at the general mathematical reader, notably the elegant and entertaining sketch in [**4**] and the more sober account of the paradigmatic case in [**6**]. Shelah's inimitable essays in this direction [**12, 13**] have a unique value.

2. The structure of classification theory. Classification theory in the first order case is essentially a trope on the notion of free amalgamation, which as analyzed by model theorists involves two quite distinct notions. Given structures $M_0 \subseteq M_1, M_2 \subseteq M_3$, one wants the following notion to be meaningful: "M_3 is the free amalgam of M_1 and M_2 over M_0." In the case of algebraically closed fields, this would mean that M_1 and M_2 are free over M_0, and that M_3 is the algebraic closure of their union. The notion of freeness, connected with transcendence degree (the key numerical invariant in this context), corresponds to a notion called "independence" (or in some context, "nonforking") in the general model theoretic setting, whereas the concept of algebraic closure is replaced by an even more general notion of "generation," referred to as the prime model over (i.e., generated by) $M_1 \cup M_2$.

The notion of independence used is probably the most subtle aspect of the whole theory, and one which is naturally encountered at the outset. The critical (and early) result was that if a first order class K does not *allow* a reasonable notion of independence, then K is a mess. There are two things that are likely to break down in setting up a notion of independence. If for example the structures in K carry linear orderings, then the notion is intrinsically asymmetrical: if a is to be independent from b one must in any case decide whether a is less than or greater than b. A class is called "stable" if it allows a reasonable, symmetric notion of independence. This notion of independence may lack finite character over the base M_0 (though it *is* customary to require a weak form of this property in the definition of stability). If the notion of independence does have finite character then the class is called "superstable." Thus one must prove at the outset that unstable, or stable but not superstable classes, fall on the nonstructure side. The theory continues in this vein: build up a structure theory by steps, proving at each stage that failure of the theory at some point produces a nonstructure theorem.

So much for independence. The next stage involves the notion of generation, or "prime" model. A prime model over a set A relative to K is one that embeds (elementarily) in all others. It is *minimal* if it does not embed properly in itself (over A). One needs prime models, and one needs them to be unique. There are two approaches at this point: restrict the classs K slightly to guarantee existence of prime models, then prove uniqueness, or else worry only about prime models over $M_1 \cup M_2$, where M_1, M_2 are free over their intersection M_0, with all M_i structures in K (rather than subsets of such structures). If one takes the first approach, restricting K, then the second is superfluous; but the result is less powerful. In any case the analysis based on the second approach (focussing on prime models over special sets) was worked out after the first edition of **[11]** and only in a special case.

The analysis so far gives one a good notion of the free amalgam of $M_1 \cup M_2$ over M_0: it is the prime model over $M_1 \cup M_2$, where M_1, M_2 are assumed to be independent over M_0. If this prime model is not always minimal over $M_1 \cup M_2$ then again one proves a nonstructure theorem. If the class K survives the inquisition this far, then every model in it can be analyzed as generated by a series of free amalgamations of small models, where the series of amalgamation operations is naturally indexed by a tree. Some would say that this already is a structure theory; however the "orthodox" formulation of what it means to have a structure theory turns out, upon examination, to be equivalent to the well-foundedness of this tree.

There is another feature of the theory which is seen quite clearly in classifying pure-injective modules. Indecomposable pure-injective modules correspond to "regular types" in the general theory. This is not the place to go into the background to this notion, which involves the theory of independence, but one is not surprised to see it used heavily in describing the factors that come into the general structure theory. For pure-injective modules, incidentally, the tree decomposition is trivial: the modules are represented as pure-injective hulls of direct sums of pure-injective indecomposables, which is approximately the same thing as prime models over free amalgamations with trivial base.

3. Some texts.

⟨⟨C'est le tissu des mots engagés dans l'oeuvre et agencés de façon à imposer un sens stable et autant que possible unique.⟩⟩ —M. Jourdain.

Each of the four levels of first order classification theory—independence (nonforking), generation (prime models), regular types, tree decomposition—builds on the previous ones, and books on the subject reflect this. The main texts available in addition to Baldwin's are [7, **8**, **11**]. Pillay gives a full account of independence and then runs through prime models and orthogonality in a convenient special case in his last forty pages. This remains a good introduction to the subject. Lascar goes a good deal farther by taking a more direct line through the subject, and another fifty pages: so in rapid but not unduly rushed succession he gives us independence, prime models in two contexts, regular types, and the decomposition in the case of a trivial tree (called by Shelah the nonmultidimensional case), with pleasant deviations into various special cases as the theory develops.

Baldwin's treatment aims at a systematic account of the theory roughly as it stood in 1980, and is in four parts corresponding exactly to the four layers of the theory. One of his aims is to provide an axiomatic account of the subject which will facilitate its transplantation to other contexts. Some of the side effects of this would probably confuse students, and the thoroughness of his coverage also makes it unsuitable for the innocent. The book is in part the English translation of **[11]** (which is ostensibly in English), but in addition Baldwin has taken great pains to incorporate a variety of approaches to the material which have developed in the meantime on three continents. (The first edition of **[11]** does not actually give the main theorem of classification theory, which was left for a subsequent journal article; Baldwin gets to the end of his version of the story.) There are full accounts of independence, prime models, regular types, and the tree decomposition for "acceptable" classes (this covers simultaneously the first order ε-saturated case and the case of arbitrary models of ω-stable theories). The last section contains some of the main applications of classification theory, including important results not covered by the other texts.

The definitive account of the modified first order incarnation of classification theory is still the idiosyncratic **[11]**, which amply repays close study. Shelah has completed the full first order case for countable theories **[12]**, and this will be in the second edition of **[11]**. All expositions other than Shelah's sidestep the "nonstructure" theorems, which rely critically on set-theoretical methods rather different from the rest of the theory, and which have been awkward to fit into an expository scheme; Baldwin does give the one nonstructure theorem for which a model-theoretic proof is known. What this means in practice is that at the outset one is told a few trenchant properties satisfied by any class K which *might* have a structure theory, with reference to Shelah, and on this basis a remarkably general theory is erected.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Ergodic theory and differentiable dynamics, by Ricardo Mañé, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 8, Springer-Verlag, Berlin, 1987, xi + 317 pp., $82.00. ISBN 0-387-15278-4

In 1931 G. D. Birkhoff published the proof of one of the most profound theorems of this century **[B]**. This theorem, which has come to be known as the Birkhoff ergodic theorem, is remarkable in several ways. It is the only recent instance which comes to mind of a single theorem giving rise to a whole new branch of mathematics. Moreover it is one of those rare theorems whose content and significance can largely be understood by nonmathematicians.

The motivation for the ergodic theorem came from the work of Boltzmann and Gibbs on statistical mechanics. The mathematical question arising from their work was under what conditions the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i(x))$$

exists and is independent of $x \in X$, where $f : X \to R$ is a real valued function on a space X and $T : X \to X$ is a transformation. This limit is the average value of the function f along the forward orbit of the transformation T.

Birkhoff's theorem concerns the case when (X, μ) is a finite measure space, f is measurable, and T is a measurable transformation for which the equation $A = T^{-1}(A)$ is never satisfied unless A has measure 0 or full measure. Such transformations are called *ergodic*. The theorem is often paraphrased by saying that for ergodic transformations the time average equals the space average. In other words if we consider the transformation T as a dynamic which occurs every unit of time, then for almost all starting points $x \in X$ the average value of the function f on the orbit of x as it evolves through time exists and is equal to $\int_X f d\mu$, the average value of the function f on the space X. Intuitively, if we consider the case when f is 1 on a measurable set A and 0 elsewhere, then this says that a typical particle x in an ergodic system will spend a proportion of its time in A equal to the proportion of the total volume in A. Except for the technical concept of measure 0 implicit in the conclusion about almost all x, this is easily explainable to a nonmathematician.