geometry of Riemann surfaces will find it an indispensable reference. The general reader will find the presentation lucid but will have to look elsewhere for the many applications of this material. Some of them can be found in references [**1**, **2**, **3**, **5**, **6**, and **7**] below.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

Limit theorems for sums of exchangeable random variables, by Robert L. Taylor, Peter Z. Daffer, and Ronald F. Patterson. Rowman and Allanheld, Totowa, New Jersey, 1985, 152 pp., $22.50. ISBN 0-8476-7435-5

Independent, identically distributed (i.i.d.) random variables form a cornerstone of theoretical statistics and the behavior of sums of such random variables constitutes a significant portion of probability theory. A natural generalization is from i.i.d. to exchangeable random variables, that is, to random variables $\{X_n, n \geq 1\}$ whose joint distributions are invariant under finite permutations. Indeed, the two notions are intertwined in a number of ways.

If n points are selected at random in the interval $[0, 1]$, then these n i.i.d. random variables (when ordered) partition the unit interval into $n + 1$ subintervals whose lengths X_i, $i = 1, \ldots, n+1$ are exchangeable random variables.

Alternatively, if N balls are cast at random into n cells labelled $1, 2, \ldots, n$ and Y_j is the number of the cell containing the jth ball, then $Y_1, \ldots, Y_N$ are i.i.d. However, if X_{ni} equals 1 or 0 according as the ith cell is or is not empty, then $\{X_{ni},\ 1 \leq i \leq n, n \geq 1\}$ is a double array which, for each n, encompasses a finite collection of exchangeable random variables. The row sum $\sum_{i=1}^{n} X_{ni}$ is, of course, the number of empty cells.

If two infinite sequences of i.i.d. random variables with common distributions F and G are selected with probabilities α and $1 - \alpha$ respectively then the resulting sequence $\{X_i,\ i = 1, 2, \dots\}$ is exchangeable, since the joint distribution of any subset of size k, namely, $\text{Prob}\{X_{i_1} < x_1, \dots, X_{i_k} < x_k\}$, equals $\alpha \prod_{i=1}^{k} F(x_i) + (1-\alpha) \prod_{i=1}^{k} G(x_i)$. This illustrates perhaps the simplest case of a fundamental result due to de Finetti (1937) that "an *infinite* exchangeable sequence is a mixture of i.i.d. sequences."

As intimated at the outset, the behavior of sums of independent random variables (not necessarily identically distributed) is of considerable interest in probability theory and in the realm of limit distributions of such sums existing results are definitive—even under the more general schema of double arrays of row-wise independent chance variables $\{X_{nk},\ k = 1, 2, \dots, k_n < \infty,\ k_n \to \infty,\ n \geq 1\}$. In the domain of strong limit theorems such as laws of large numbers, iterated logarithm laws, etc., known results are pervasive, especially in the important i.i.d. case. In the last decades, partial but significant extensions to Banach-space-valued random variables have been obtained.

Naturally, one would like to create a corresponding body of results covering exchangeable random variables both in the single sequence case and that of double arrays of row-wise exchangeable random variables (recall the prior example of casting balls into cells). However, while de Finetti's theorem may be exploited in the single sequence case, this is no longer true for double arrays since the finitely many exchangeable random variables in each row cannot, in general, be embedded in an infinite sequence of exchangeable random variables. Alternative martingale techniques have been employed to advantage but the ensuing theorems do not subsume the simple "balls into cells" example for which it is known, via ad hoc methods, that under suitable normalization and a natural hypothesis, the number of empty cells has a limiting normal distribution. Even in the single sequence case, results are fragmentary and so the totality of limit theorems for exchangeable random variables remains a small fraction of that for independent random variables. Thus, any hopes for a comprehensive or cohesive survey of this subject must await considerable additional research and most likely the next century.

The current monograph focuses on limit theorems for weighted averages $\sum_{k=1}^{k_n} a_{nk} X_{nk}$ of row-wise exchangeable random variables $\{X_{nk},\ 1 \leq k \leq m_n,\ n \geq 1\}$ with $\{a_{nk},\ 1 \leq k \leq k_n \to \infty,\ n \geq 1\}$ an array of constants, thereby generalizing known results for $\sum_{k=1}^{k_n} X_{nk}$. As in earlier work, the cases $m_n \equiv \infty$, $n \geq 1$, and $k_n \leq m_n < \infty$, $n \geq 1$, require separate treatment. The first three chapters, which cover real random variables, deal respectively with preliminaries, Central Limit Theorems and Laws of Large Numbers while the last three follow the same format for Banach space random variables.

The initial chapter sets forth elementary properties of exchangeable random variables and contains an instructive example of a nonintegrable exchangeable sequence obeying the Strong Law of Large Numbers.

Chapter 2 provides conditions under which $\sum_{k=1}^{k_n} a_{nk} X_k$ or $\sum_{k=1}^{k_n} a_{nk} X_{nk}$ converges in distribution to a scale parameter mixture of normal distributions. Here, a martingale limit theorem and the reverse martingale approach of Weber are the main tools. The cited Theorem 2.2.4 of Weber [**5**] constitutes an improvement of an earlier result of Blum, Chernoff, Rosenblatt and Teicher [**1**] only when the condition $EX_{n1}^4 < \infty$, $n \geq 1$, is deleted (which is permissible as noted by Weber). In the proof of Theorem 2.1.2, it is not the case $r = 1$ of

Lemma 2.1.1 that is needed but rather the more general fact [**1**] that if EX_1X_2 exists and $\{X_n, n \geq 1\}$ is an $\mathcal{L}_1$ exchangeable sequence then $\mathrm{Cov}(X_1, X_2) = 0$ iff $E_{\mathcal{F}}X_1$ is almost surely constant.

Chapter 3 contains a motley collection of laws of large numbers including one by Edgar and Sucheston [**3**] for $\mathcal{L}_p$ exchangeable random variables, $0 < p < 1$. The subsequent Theorem 3.2.8 has nothing to do with exchangeability and is an immediate consequence of the fact that if $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} |a_n X_n|$ converges almost surely for any sequence of random variables satisfying $\sup_{n\geq 1} E|X_n| < \infty$. Proposition 3.3.4 holds even when the basic random variables $\{Y_i,\ 1 \leq i \leq n\}$ are exchangeable rather than i.i.d. while Proposition 3.3.5 follows directly from the Strong Law for i.i.d. random variables in view of

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \overline{Y}_n^2 \overset{\text{a.s.}}{\to} EY_1^2 - (EY_1)^2.$$

Chapter 4 considers random elements on a separable, real Banach space E and paves the way for subsequent chapters by discussing a version of de Finetti's theorem, geometric properties of Banach spaces such as type p, uniformly smooth and B-convex, etc. An example of Chatterji for which the martingale convergence theorem fails is mentioned but not referenced.

In Chapter 5 the result of Hoffman-Jørgensen and Pisier [**4**] (that i.i.d. random elements on E with mean 0 and finite second moment obey the CLT iff E is of type 2) is generalized to weakly uncorrelated ($E\|X_n\|^2 < \infty$, $n \geq 1$, $\mathrm{Cov}[f(X_i), f(X_j)] = 0$, $i \neq j$, all $f \in E^*$) exchangeable sequences $|X_n, n \geq 1|$, the limit measure again being a mixture of normals. This, in turn, is extended to linear combinations $\sum_{k=1}^{k_n} a_{nk}X_k$. In the case of double arrays of rowwise exchangeable Banach-space-valued random variables $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$, utilizing martingale methods and centerings by conditional expectations Z_{nk}, conditions are given for weak convergence of $\sum_{k=1}^{k_n} a_{nk}(X_{nk} - Z_{nk})$.

The concluding Chapter 6 provides conditions under which $\|\sum a_{nk}X_{nk}\| \to 0$ in probability, almost surely and completely. Examples and counterexamples in this and the preceding chapter would have given the reader a better appreciation of the desirability and necessity of the varying hypotheses.

This monograph in conjunction with the notes of David Aldous [**2**] on *Exchangeability and related topics* should serve to renew interest in a topic that well merits it.

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