BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 \$1.00 + \$.25 per page

Representation theory of semisimple groups. An overview based on examples, by Anthony W. Knapp. Princeton Mathematics Series, vol. 36, Princeton University Press, Princeton, N. J., 1986, xvii + 773 pp., \$75.00. ISBN 0-691-08401-7

A group representation is a homomorphism π of a group G into the group of linear transformations of a vector space V. It is called *irreducible* if V is not zero, and no proper subspace of V is preserved by the operators in $\pi(G)$. It is called *unitary* if V carries a Hilbert space structure preserved by $\pi(G)$. Unless otherwise stated, I will assume that all representations considered are unitary. (I will also suppress many extremely important technical qualifications throughout. The reader wishing a slightly more accurate picture may assume that all groups are locally compact, separable, and type I.) Representations of finite groups were first investigated by Frobenius and Schur. Hermann Weyl in the 1920s extended many of their ideas to compact groups. Weyl's work found its way into quantum mechanics. Because the Poincaré group P (of automorphisms of a flat space-time preserving the notion of interval) is not compact, relativity required considering representations of noncompact groups. Wigner in 1939 analyzed most of the irreducible representations of P, by a clever use of the normal subgroup T of translations (see **[15]**). (T is isomorphic to $\mathbf{R}^4$ and its representations are relatively easy to understand.) In fact Wigner found everything which was of physical interest to him. The remaining irreducible representations he described in terms of the (unknown) irreducible representations of the groups $\mathrm{SL}(2, \mathbf{R})$ and $\mathrm{SL}(2, \mathbf{C})$. (These are the groups of two by two matrices of determinant one over the real and complex fields.)

Wigner had found a fundamental division of representation theory into two parts. His ideas were greatly extended by Mackey, becoming what is now called the "Mackey machine": a systematic method for describing the irreducible representations of a group in terms of those of a normal subgroup. The centerpiece of Mackey's method is the idea of induced representations. Suppose H is a closed subgroup of G. To any representation (π, V_π) (even infinite-dimensional) of H, one can associate a vector bundle $\mathscr{V}_\pi$ over the homogeneous space G/H. The fiber of this vector bundle over the identity coset eH is the original space V_π. The action of G on G/H lifts to $\mathscr{V}_\pi$, and G therefore acts on the vector space of sections. This action Π is called the *representation induced by* π:

$$\Pi = \mathrm{Ind}_H^G(\pi).$$

Roughly speaking, the Mackey machine shows how to exhibit many irreducible representations as induced from proper subgroups. That is, it realizes the representations in an elegant geometric way: as spaces of sections of vector bundles. Mackey's theory has been carried very far. Its only serious limitation is that one needs a good normal subgroup of G to show that all irreducible representations actually appear. This is the first part of representation theory, and the part most clearly foreshadowed in Wigner's work.

The papers **[2]** and **[6]**, published in 1947, began the second part of representation theory. They also have their roots in **[15]**, for they complete the determination of the irreducible representations of P begun by Wigner: Bargmann found all the irreducible representations of $SL(2, \mathbf{R})$, and Gelfand and Naimark found those of $SL(2, \mathbf{C})$. Yet they represent almost a complete departure from Wigner's work in their methods, for these groups have no proper normal subgroups except for their two-element centers. A Lie group is called *simple* if every proper normal subgroup of it is finite; it is *semisimple* if it is (up to finite covering) a direct product of simple groups. The second part of representation theory is the part concerned with semisimple groups; it is of course the subject of the book under review. It is best described not from the perspective of these two papers, but rather by taking full advantage of hindsight.

What semisimple groups lack in normal subgroups they make up in almost every other respect. There are very few of them (only finitely many of any one dimension), and they have an incredibly rich and rigid structure. The prototypical example is $SL(n, \mathbf{R})$, the group of n by n matrices of determinant one. Matrices may be subjected to row reduction, the Gram-Schmidt process, polar decomposition, or Jordan decomposition; all of these processes have analogues for the elements of a general semisimple group G. $SL(n, \mathbf{R})$ acts on the space of positive definite quadratic forms on $\mathbf{R}^n$, and on each of the Grassmannian manifolds of k-planes in $\mathbf{R}^n$; analogues of these homogeneous spaces exist for all semisimple groups. These are some of the basic tools of representation theory for semisimple groups. Their familiar nature gives to the subject a kind of particularity which is not common in serious mathematics. Most of the objects of interest come in countable families and can be treated almost as individuals. The established wisdom (coming largely from Harish-Chandra) is that one ought *not* to treat them so, and certainly Knapp's book seeks to convey general methods. Nevertheless, the fact that few examples exist makes each example much more interesting. It is part of the reason that the book succeeds.

Mackey's work in the presence of normal subgroups at least suggests a source of some irreducible representations of a semisimple group G: induced representations. Recall that such a representation lives on the space of sections of a vector bundle on a homogeneous space G/P. Early results of Gelfand and Naimark **[7]** showed that the appropriate homogeneous spaces are what are now called *generalized flag manifolds*; these include the Grassmann manifolds and the ordinary flag manifold in the case of $SL(n, \mathbf{R})$. Equivalently, the isotropy group P is required to be *parabolic*. In addition, the inducing representation π of the isotropy group P should be trivial on the commutator subgroup of the solvable radical of P. When both of these conditions are met, we will call $\operatorname{Ind}_P^G(\pi)$ a *parabolically induced* representation. It is likely that other induced representations are essentially never irreducible, but I do not know whether this has been proved.

The first question that must be addressed is the irreducibility of parabolically induced representations. Bruhat in **[4]** recognized that the irreducibility was related to certain natural maps among the representations, the *standard intertwining operators*. (An intertwining operator is just a linear map between the spaces of two different representations of the same group that respects the group actions. For semisimple groups the term often refers particularly to these standard intertwining operators, which have a long and colorful history—in fact several colorful histories among which one can choose.) Construction of the standard intertwining operators is somewhat difficult, but it yields unexpected rewards. Harish-Chandra showed that these operators control the way the parabolically induced representations fit into harmonic analysis on G (understood here as the decomposition of $L^2(G)$ under the action of G). They are also central to the theory of complementary series representations, which I will not discuss at all.

We turn now to the irreducible representations that are not parabolically induced. The most interesting of these are the *discrete series*, which are the irreducible subrepresentations of $L^2(G)$. Write X for the symmetric space of maximal compact subgroups of G; for $\mathrm{SL}(n, \mathbf{R})$ this is just the space of positive definite quadratic forms in n variables. In general X is a complete Riemannian manifold. Suppose $\mathscr{V}$ is a homogeneous Hermitian vector bundle on X. Write $L^2(X, \mathscr{V})$ for the space of square-integrable sections of $\mathscr{V}$. There is a natural G-invariant second-order elliptic operator L on sections of $\mathscr{V}$. Each eigenspace of L on $L^2(X, \mathscr{V})$ turns out to be a direct sum of a finite number of discrete series representations. Constructing discrete series representations amounts to constructing eigenfunctions of L that decay nicely at infinity on X. Such functions can now be found in several ways. Harish-Chandra's original approach in **[8]** and **[9]** remains awe-inspiring, and it provides a wealth of auxilary information about harmonic analysis on G. Nearly fifteen years elapsed before Atiyah and Schmid **[1]** and Flensted-Jensen **[5]** found simpler arguments.

There remain the noninduced representations that do not occur as subrepresentations of $L^2(G)$. These are far from completely understood. Their analysis begins with the Langlands classification, which provides an explicitly parametrized list of all (possibly nonunitary) irreducible representations of G. Its shortcomings are that the representations are not very explicitly constructed, and that it is difficult to determine which of them admit unitary structures. Current research is directed at the resolution of problems like these.

These are the highlights of representation theory for semisimple groups as it existed by about 1975: parabolic induction, the discrete series, and the Langlands classification. They are also the main themes in Knapp's book. Each is developed carefully and thoroughly, with beautifully worked examples and proofs that reflect long experience in teaching and research. There are even problems (almost unheard of in a book at this level). Graduate students can read it unaided, and I suspect that one could teach several good courses from it. An appendix summarizes in detail what one needs from the general theory of Lie groups; any first course would at least make this appendix very easy to read.

The only other books of roughly comparable scope are those of Warner [**13**]. ([**3**], [**11**], and [**12**] are narrower and technically more demanding.) Warner's books were written before the Langlands classification was established, so they have a little less to do; yet Knapp's book is shorter. Knapp achieves this miracle in two ways. First, he takes advantage of simplifications in Harish-Chandra's original proofs. For example, he presents Flensted-Jensen's construction of the discrete series, and sketches Herb's proof [**10**] of Harish-Chandra's Plancherel formula. Most importantly, however, he simply omits some proofs. This is always done with ample notice to the reader, detailed references, and complete treatment of illuminating special cases. The result is delightful: a readable text which loses almost none of its value as a reference work.

Finally, there is the question of what topics have been omitted entirely. Most experts would agree that a fourth topic should now be added to the basic three already mentioned. Regrettably they would not agree on what that fourth topic is. Some candidates (listed more or less alphabetically) are non-Riemannian symmetric spaces; Beilinson-Bernstein localization; the Kazhdan-Lusztig conjectures; and Zuckerman's cohomological parabolic induction. The reader of Knapp's book will be well equipped to pursue the first of these, but the rest proceed in rather different directions. The problem is not with the book, but with a genuine (if incomplete) division of the subject into analysis and algebra. Knapp is an analyst; he uses all the algebra he needs to smooth the reader's path, but the destinations chosen betray his taste. As that taste is excellent, however, even those handicapped by a preference for algebra should relax and enjoy the ride.

REFERENCES

1. M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Invent. Math. **42** (1977), 1–62.

2. V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48** (1947), 568–640.

3. A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton Univ. Press, Princeton, N.J., 1980.

4. F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France **84** (1956), 97–205.

5. M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. (2) **111** (1980), 253–311.

6. I. M. Gelfand and M. A. Naimark, *Unitary representations of the Lorentz group*, Izv. Akad. Nauk SSSR **11** (1947), 411–504.

7. ______, *Unitary representations of the classical groups*, Trudy Mat. Inst. Steklov 36, Moscow-Leningrad, 1950. German translation: Akademie-Verlag, Berlin, 1957.

8. Harish-Chandra, *Discrete series for semi-simple Lie groups*. I, *Construction of invariant eigendistributions*, Acta Math. **113** (1965), 241–318.

9. ______, *Discrete series for semi-simple Lie groups*. II, *Explicit determination of the characters*, Acta Math. **116** (1966), 1–111.

10. R. Herb, *Fourier inversion and the Plancherel theorem for semisimple real Lie groups*, Amer. J. Math. **104** (1982), 9–58.

11. V. S. Varadarajan, *Harmonic analysis on real reductive groups*, Lecture Notes in Math., vol 576, Springer-Verlag, Berlin and New York, 1977.

12. D. Vogan, *Representations of real reductive Lie groups*, Birkhäuser, Boston-Basel-Stuttgart, 1981.

13. G. Warner, *Harmonic analysis on semisimple Lie groups*, vols. I and II, Springer-Verlag, Berlin, and New York, 1972.

14. H. Weyl, *Theorie der Darstellung kontinuierlicher Gruppen*. I, II, III, Math. Z. **23** (1925), 271–309; **24** (1925), 328–376, 377–395.

15. E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. (2) **40** (1939), 149–204.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Localization in Noetherian rings, by A. V. Jategaonkar, London Mathematical Society Lecture Note Series, vol. 98, Cambridge University Press, Cambridge, 1986, xii + 324 pp., $34.50. ISBN 0-521-31713-4

The book under review is, to start with, a book about noncommutative Noetherian rings and prime ideals. (We call a ring Noetherian if it satisfies the ascending chain condition on both left and right ideals.) We should first ask why the Noetherian condition and prime ideals are of significance in the noncommutative case. Commutative Noetherian rings arise naturally, for example as the coordinate rings of algebraic varieties, and the prime ideals then correspond to irreducible subvarieties. Noncommutative rings arise most naturally as rings of operators of some sort—some of the first ones studied in this century were rings of differential operators. It is therefore not surprising that the primitive ideals (annihilators of irreducible representations) were studied before prime ideals, and that Artinian rings were studied before Noetherian rings. Jacobson's definitive book of 1956 makes no mention of the Noetherian condition. However, Noetherian rings do arise naturally—certain rings of differential operators are Noetherian, in particular, the enveloping algebras of finite-dimensional Lie algebras. Also, group algebras of polycyclic-by-finite groups are Noetherian. (A group is polycyclic-by-finite if it has a series of normal subgroups such that all of the factors are finite or finitely generated Abelian.) This suggested that the development of a theory making full use of the Noetherian hypothesis should shed particular light on some of the most important examples in representation theory, and recent events have borne this out. The correct definition of prime ideal was pointed out by Krull in 1928 (an ideal P is prime if for any pair of ideals A and B, if $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$). (Prime ideals force themselves to one's attention, even if one is only really interested in irreducible representations and primitive ideals.) However, the first important theorem about prime ideals in the Noetherian case was Goldie's theorem, which was not proved until 1958.

What Goldie's theorem provides is the correct analogue in the noncommutative case for the usual field of fractions of a domain in the commutative case. If