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Nonstandard methods in stochastic analysis and mathematical physics, by Sergio Albeverio, Raphael Høegh-Krohn, Jens Erik Fenstad, and Tom Lindstrøm, Academic Press, Orlando, xi + 514 pp., $59.50 cloth, $39.50 paperback. ISBN 0-12-048860-4

Nonstandard or infinitesimal analysis was invented by the late Abraham Robinson in 1960. Since that time there has been continued interest in the subject and a number of impressive results have been established using nonstandard methods. These results testify to the vision of the man of whom Gödel wrote, "(He was) the one mathematical logician who accomplished incomparably more than anybody else in making this science fruitful for mathematics. I am sure his name will be remembered by mathematicians for centuries." The book under review is a welcome addition to a growing list devoted to the subject.

Nonstandard analysis has had a controversial history. It had its roots in the use of infinitesimals by Leibniz and Newton in the development of calculus. Infinitesimals are "numbers" which are smaller in absolute value than any real number. Leibniz regarded them as entities in some "ideal" structure which also contained the infinitely large numbers and the reals. He also implicitly made the important but somewhat vague hypothesis that this structure satisfied the same rules as the ordinary real number system. The challenge facing Robinson was thus to

(a) demonstrate the existence of a set $*R$, now called the hyperreal numbers, which carried analogues of all the structures on the reals R (for example, the ring and the set theoretic structures);

(b) ensure that statements true in the real number system are mirrored in a natural way by statements true in the structures on $*R$.

The statement (b) is a special case of what is now known as the transfer principle. This principle has turned out to be the central tool in nonstandard analysis. To deal with it, a little mathematical logic is necessary in order to make precise what is meant by the words "statement" and "true". But it is important to know that in practice nonstandard analysis involves only a minimal amount of mathematical logic. Indeed, it entails essentially none once the foundations have been established. Robinson himself used mathematical logic (in particular, the compactness theorem of mathematical logic) to establish his results. He then went on to show how these ideas could be used to make infinitesimal methods in calculus rigorous, and observed that many standard results involving various types of limits often had easier nonstandard proofs. If Robinson had achieved only this vindication of Leibniz, nonstandard analysis would be just a footnote to history. But he realized that every mathematical structure has a nonstandard model which can be used to gain knowledge of the original structure by application of the appropriate transfer principle. This led to the great variety of applications that we see today.

Robinson's book *Nonstandard analysis* **[15]** appeared in 1966 and set the stage for a much more intensive development of the subject. However, his proof of the basic theorems on nonstandard models involved a difficult excursion into mathematical logic and perhaps persuaded many nonlogicians that they could never master the subject. Fortunately it turns out that nonstandard models can be constructed as ultrapowers, and the transfer principle follows from Łoš' theorem. This method, which is used in the book under review, is much more accessible to nonlogicians. Thus nonstandard analysis can be regarded as ultrapower methods made easy by systematic use of the transfer principle. Many nonstandard arguments derive additional strength by using κ-saturated models, a notion which was extensively investigated in Luxemburg's paper *A general theory of monads*, which appeared in **[10]**. (Saturation is available in the ultrapower setting if the ultrapowers are constructed from κ-good ultrafilters.)

The first part of the book under review, called the Basic Course, develops all of the material needed later. After presenting the foundations, the first chapter continues with a nonstandard development of the calculus, including Robinson's **[14]** simple proof of the Peano existence theorem for ordinary differential equations, which avoids the Arzela-Ascoli theorem, along with elaborations due to Birkeland and Normann. The chapter also contains an introduction to some interesting work on singular perturbation theory for ordinary differential equations due to Callot and F. and M. Diener.

After discussing saturation, the second chapter presents some applications to topology and functional analysis, including a brief discussion of applications to Banach space theory by Henson and Moore (see the survey in **[5]**). Related work using ultrapower techniques (without nonstandard analysis) was initiated by Dacunha-Castelle and Krivine and is now extensively employed. But the nonstandard theory has conceptual advantages over the corresponding ultrafilter approach, since it uses natural tools and concepts which have no simple counterpart in the ultrapower setting.

A large part of the book under review is devoted to applications of nonstandard analysis to various aspects of probability theory, and the foundations are presented in Chapter three. This work began with Peter Loeb's important paper [8] on nonstandard measure theory. Until that time, measure theory had posed problems to nonstandard analysts because there was no way of dealing with countable additivity in the internal structure of the nonstandard model. Loeb's idea was to make nonstandard finitely additive premeasure spaces into standard measure spaces (known now as Loeb spaces). He used this idea to give nonstandard representations of several problems in continuous probability, in particular the Poisson process. This work paved the way for Robert Anderson [1] to represent Brownian motion as an infinitesimal random walk. The book under review briefly mentions the work of Edwin Perkins, who used the same ideas to settle an important conjecture in Brownian local time in his Ph.D. thesis [12]. A central theme in these applications is that problems in probability of a continuous nature can be represented by discrete problems in hyperfinite nonstandard models. These models are finite from the nonstandard, but infinite from the standard point of view. Their use makes for an intuitive clarity and ease of calculation which can simplify difficult problems.

The second part of the book begins with a long chapter on applications of nonstandard analysis to stochastic processes which explores at length the ideas mentioned in the previous paragraph. In the last few years the number of applications has exploded, and any list is necessarily incomplete. The following should at least be mentioned, since they are covered in the book: Itô integration (Anderson), stochastic integration with respect to martingales (Lindstrøm, Hoover, and Perkins), very general existence theorems for stochastic differential equations (mostly due to Keisler and his students), optimal control for stochastic differential equations (Nigel Cutland), stochastic integration in infinite-dimensional spaces (Lindstrøm), and Levy Brownian motion (A. Stoll).

One of the great merits of the book is that it presents many applications of nonstandard analysis to mathematical physics. Two of the authors, Albeverio and Høegh-Krohn, have done extensive research in quantum field theory and related areas, and some of their work appears here for the first time. There is a chapter on hyperfinite Dirichlet forms and Markov processes, in which hyperfinite linear algebra and Markov chains replace the usual highly technical approach using functional analysis and continuous-time Markov processes. The authors expect that the easier nonstandard methods will lead to solutions of infinite-dimensional problems in quantum field theory and hydrodynamics.

Next, there is a chapter devoted to various topics in differential operators. The first section deals with an eigenvalue problem for a Sturm-Liouville equation with a measurable coefficient by using a discrete approximation (Birkeland, Normann). In the second section, singular perturbations of nonnegative operators are investigated and applied to differential operators with potentials concentrated in points and along Brownian paths, with applications to polymer models. The final section is an introduction to recent work on the Boltzmann equation by Leif Arkeryd, who has just obtained the most general existence theorem so far known for initial conditions far from equilibrium. His strategy is to go to a nonstandard model and use transfer of known existence results for truncated problems when the truncation parameters are infinite.

The last chapter deals with hyperfinite lattice models. First the authors discuss stochastic evolution and classical equilibrium theory for hyperfinite lattices with finite spacing, the nonstandard equivalent of the corresponding theories for infinite particle systems. Next comes a section on the global Markov property, which includes results due to Kessler, and which has applications to quantum field theory. In the last sections, the authors model Euclidean quantum field theory as a continuous spin system on a hyperfinite lattice with infinitesimal spacing, and discuss connections with polymer measures.

This excellent book is an encyclopedic survey of the applications of nonstandard analysis to the areas we have discussed. But even so, it only scratches the surface of possible applications of nonstandard analysis. The technique has produced new results in areas as diverse as complex function theory (Robinson, Behrens), potential theory (Loeb), number theory (Robinson, Roquette), and mathematical economics (Robinson, Brown, Anderson, etc.), to mention a few not covered in the text. In fact, a number of important results, for example in probability theory and the theory of Banach spaces, have so far only been proved using ultrapower or nonstandard techniques (for a discussion of the strength of nonstandard methods, see [3]). As more people realize its latent power, nonstandard analysis will likely become an integral part of most mathematicians' working vocabulary. Anyone whose interest has been aroused can find more references in the books listed below.

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A local spectral theorem for closed operators, by Ivan Erdelyi and Wang Shengwang. London Mathematical Society Lecture Notes Series, vol. 105, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985, $22.95. ISBN 0-521-31314-7

Spectral theory has its origin in the theories of matrices and integral equations, and in the case of unbounded operators, in the theory of differential equations. David Hilbert was the first mathematician to use the word "spectrum" in its present meaning: if T is a bounded operator acting in a (complex) Banach space X, its spectrum $\sigma(T)$ is the set of all complex numbers λ, for which $\lambda I - T$ is singular (that is, not invertible in the Banach algebra $B(X)$ of all bounded linear operators on X). A central role in the study of the spectrum is played by the resolvent operator $R(\lambda:T) = (\lambda I - T)^{-1}$, defined and analytic in the resolvent set $\rho(T) = \mathbf{C} \setminus \sigma(T)$, and by the analytic operational calculus τ: $f \to f(T)$, which is a continuous homomorphism of the topological algebra $H(\sigma(T))$ of all functions analytic in a neighborhood of $\sigma(T)$ into $B(X)$, such that $\tau(f_0) = T$ for $f_0(\lambda) \equiv \lambda$. The theory of analytic functions became a powerful tool in spectral theory in the work of Laguerre (1867), Frobenius (1896), and Poincaré (1899), and later in the now classical work of F. Riesz, Hilbert, Wiener, Stone, Beurling, Gelfand, Dunford, and many others. The related spectral theory of Banach algebras, pioneered by Gelfand and Shilov, follows a similar path. For unbounded closed operators, Taylor extended the Riesz-Dunford integral formula for $\tau(f)$ in the form

$$\tau(f) = f(\infty) + \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda;\, T)\,d\lambda$$

for f analytic in a neighborhood of $\sigma(T) \cup \{\infty\}$ (and Γ a suitable contour).

The analytic operational calculus leads to the construction of projections $E(\sigma)$ associated with the spectral sets σ of T (i.e., the open-closed subsets of $\sigma(T)$ in the relative topology), by means of the formula

$$E(\sigma) = \frac{1}{2\pi i}\int_{\Gamma(\sigma)} R(\lambda;\, T)\,d\lambda$$