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Completely bounded maps and dilations, by Vern I. Paulsen. Pitman Research Notes in Mathematics, vol. 146, Longman Scientific and Technical, Essex, and John Wiley and Sons, New York, 1986, 187 pp., $38.95. ISBN 0-582-98896-9

This monograph uses C^*-algebraic techniques to study operator-theoretic problems. In particular, it uses the theory of completely positive and completely contractive maps to study dilations of operators. If T is a bounded linear operator on a Hilbert space H, then a dilation of T is a bounded linear operator S on a Hilbert space K containing H such that $Tx = PSx$ for each x in H, where P is the orthogonal projection of K onto H. Sometimes information about T can be obtained by dilating T to a "nice" operator S, using known facts about S, and then compressing back to H. Let $L(H)$ denote the algebra of all bounded linear operators on Hilbert space H. The two earliest dilation theorems are due to Naimark [3] and Sz.-Nagy [5]. Naimark proved that a regular, positive, $L(H)$-valued measure on a compact Hausdorff space can be dilated to be a spectral measure. Sz.-Nagy proved that if T belongs to $L(H)$ with the norm of T less than or equal to one, then T can be dilated to a unitary U such that $T^n x = PU^n x$ for all $n \geqslant 1$ and all x in H. Sz.-Nagy used this to prove von Neumann's inequality: If the norm of T is less than or equal to one and p is a polynomial, then $\|p(T)\| \leqslant \|p\|_\infty$, where $\|p\|_\infty$ denotes the uniform norm of p on the unit circle. Dilation theorems of various types are now standard in operator theory. See the book of Foiaş and Sz.-Nagy [6] or Halmos' Problem Book [2].

In 1955 W. F. Stinespring introduced a C^*-algebraic approach to dilation theory and used it to prove Naimark's theorem [4]. Besides the applications to dilation theory, Stinespring's approach turned out to be fundamental for the general theory of C^*-algebras. The key idea was to realize that if A is a C^*-algebra, then $M_n(A)$, the algebra of n-by-n matrices over A, is also a C^*-algebra. A C^*-algebra carries along with it all the information in the $M_n(A)$. An element of a C^*-algebra is called positive if it is of the form $a*a$, and a linear map from one C^*-algebra to another is called positive if it takes positive elements to positive elements. For f a linear map from a C^*-algebra A to a C^*-algebra B, Stinespring defined f_n from $M_n(A)$ to $M_n(B)$ by $f_n(a_{ij}) = (f(a_{ij}))$ and called f completely positive if each f_n was positive. Not all positive linear maps are completely positive. Stinespring proved that if f: $A \to L(H)$ is completely positive, then there is a Hilbert space K, a *-homomorphism π: $A \to L(K)$, and a bounded linear transformation V from H to K such that $f(a) = V*\pi(a)V$ for all a in A. Stinespring proved that if A is commutative, then every positive linear map on A is completely positive. This implies Naimark's dilation theorem.

If f is a linear map from a C^*-algebra A to a C^*-algebra B, f is called completely contractive if the norm of f_n is less than or equal to one for each n. Although the idea of complete positivity originated with Stinespring in 1955, the concept did not receive much attention until the 1969 paper of Arveson [**1**], who introduced complete contractivity and proved a Hahn-Banach type extension theorem for completely positive linear maps.

The book under review develops the Stinespring-Arveson theory as well as the more recent work of Wittstock and others (including Paulsen). These C^*-algebraic results are applied to a variety of operator-theoretic questions, including the dilation theorems of Naimark, Sz.-Nagy, and Berger, as well as the theory of spectral sets. Each of the ten chapters concludes with a number of carefully chosen exercises. Although many of the exercises are results from research papers, the development is such that the exercises should be within the reach of students who start the book with a modest knowledge of C^*-algebras and operator theory. The book is a testimony to how much can be done with relatively elementary methods if one completely understands the situation. I highly recommend the book to all those interested in the interplay between C^*-algebras and operator theory.

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