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THE TRACE FORMULA FOR VECTOR BUNDLES

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Let X be a compact Riemannian manifold and let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the spectrum of the Laplace operator. By a theorem of Hermann Weyl the spectral counting function

$$N(\lambda) = \#\{\lambda_i^2 < \lambda\} \tag{I}$$

satisfies a growth estimate of the form $O(\lambda^N)$, so its Fourier-Stieltjes transform

$$\int e^{i\lambda t}\, dN(\lambda) = \sum_k e^{\pm i\sqrt{\lambda_k t}} \tag{II}$$

is a tempered distributional function of t. The classical trace formula says that the singular support of (II) is contained in the length spectrum of X. Moreover, under suitable hypotheses on geodesic flow, the trace formula gives considerable information about the singularities in (II). (See [**DG** and **C**].)

There is a fairly straightforward (and not terribly interesting) generalization of the trace formula to vector bundles. (See, for instance, the introduction to [**DG**].) We will be concerned in this article with a much more subtle generalization inspired by recent articles of Hogreve, Potthoff, and Schrader [**HPS**], and Schrader and Taylor [**ST**] in *Communications in Mathematical Physics.*

Let G be a compact Lie group and $\pi\colon P \to X$ a principle G-bundle with connection. Given a finite-dimensional unitary representation, ρ, of G we will denote by $E\rho$ the vector bundle over X associated with ρ and by D_ρ the associated connection.

Now consider a *ladder* $\{\rho_e,\ e = 1, 2, \ldots\}$ of irreducible representations of G. (This means that the maximal weight of ρ_e is e times the maximal weight of ρ_1.) For given e let

$$\lambda_{k,e}, \qquad k = 1, 2, 3, \ldots$$

be the spectrum of the Laplace operator on $C^\infty(E\rho_e)$:

$$\Delta_e = D^*\rho_e D\rho_e + e^2.$$

The Hogreve-Potthoff-Schrader and Schrader-Taylor papers are concerned with asymptotic properties of the quantities e and $E = \lambda_{k,e}$ when e and E tend to infinity in such a way that the ratio $r = e\colon \sqrt{E}$ is (approximately) constant. One way to measure such asymptotic behavior is as follows: Fix a Schwartz function of one variable, $\varphi(s)$, with $\varphi(s) \geq 0$ and $\int \varphi(s)\, ds = 1$, and form the sum

$$N_{\varphi,r}(\lambda) = \sum_e \sum_{\lambda_{k,e} < \lambda^2} \varphi(r\sqrt{\lambda_{k,e}} - e). \tag{III}$$

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This sum is similar to the sum (I) except that it counts eigenvalues in such a way that eigenvalues for which the ratio $e\colon \sqrt{E}$ is close to r are weighted much more heavily than eigenvalues for which the ratio is far away from r. Our main result is a trace formula for the Fourier-Stieltjes transform of $N_{\varphi,r}$. To formulate it we need to recall some facts about the coupling of a classical dynamical system to a Yang-Mills field. Let O_e be the co-adjoint orbit in g^* associated with the representation, ρ_e, and let M_e be the symplectic manifold obtained by reducing T^*P with respect to O_e. By a theorem of Sternberg [**S**] and Weinstein [**W**] the choice of a connection on P gives rise to a symplectic fiber mapping

$$\pi_e\colon M_e \to T^*X.$$

If H is the standard Kinetic energy Hamiltonian on T^*X describing the motion of a classical particle on X when no background field is present, $\pi_e{}^*H$ describes the motion of a classical particle (of "charge" e) when a background field is present. (See [**S**].)

Now fix e and E so that $e\colon \sqrt{E} = r$ and consider the restriction to the energy surface $\pi_e{}^*H = E$ of the Hamiltonian system on M_e associated with $\pi_e{}^*H$. We will assume that the periodic trajectories of this system are *non-degenerate* and denote the set of these trajectories by Γ.

THEOREM. *If the function $\hat{\varphi}$ has compact support, the Fourier-Stieltjes transform of $N_{\varphi,r}(\lambda)$ can be written as a locally finite sum*

$$e_0(t) + \sum_{\gamma\in\Gamma} e_\gamma(t), \tag{IV}$$

where e_0 and the e_γ's are distributions of compact support. Moreover, e_0 has no singularities except for a classical conormal singularity at the origin, and e_γ has no singularities except for a classical conormal singularity at $t = T_\gamma + r\theta_\gamma$. Here T_γ is the period of γ and θ_γ an appropriate determination of

$$\frac{\mathrm{Log}(\textit{holonomy of } \gamma)}{2\pi i} - rT_\gamma. \tag{V}$$

Moreover, $e_\gamma(t)$ is equal to

$$c_\gamma \delta(t - T_\gamma - r\theta\gamma) + \tilde{e}_\gamma(t) \tag{VI}$$

where $\tilde{e}_\gamma(t)$ is in $\mathcal{L}^1$ and the constant c_γ is related to the primitive period T_γ^* of γ and the linearized Poincaré map, P_γ, by the formula

$$c_\gamma = \hat{\varphi}(\theta_\gamma)\frac{T_\gamma^*}{2\pi}\frac{\exp(i\pi\sigma_\gamma/4)}{|I - P_\gamma|^{1/2}}, \tag{VII}$$

σ_γ being a suitable Maslov index.

REMARKS. (1) There is an analogue of (VI) for $e_0(t)$ as well, which we won't both to describe here. From it one gets a Weyl formula for the asymptotic behavior of $N_{\varphi,r}(\lambda)$.

(2) Just as in the classical case, an analogue of the expansion (V) is true when the periodic trajectories on the $\pi_e{}^*H = E$ energy surface form "clean" submanifolds. (See [**DG**, §6].)

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