BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 1, July 1986

0273-0979/86 $1.00 + $.25 per page

Approximation of Hilbert space operators, Volume I, by Domingo Herrero, Pitman Publishing Inc., Boston, 1982, xiii + 255 pp., $23.95. ISBN 0-273-08579-4

Approximation of Hilbert space operators, Volume II, by Constantin Apostol, Lawrence Fialkow, Domingo Herrero and Dan Voiculescu, Pitman Publishing Inc., Boston, 1984, x + 524 pp., $29.95. ISBN 0-273-08641-3

1. Introduction. The theme of the books under review is *approximation*; that is, how well do simple models of operators approximate larger, less understood classes? This usually means approximation in the operator norm, but it may well ask for more. For example, one might ask that the error of estimation be compact. As the set of compact operators is the only proper, closed, two-sided ideal in the algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a separable Hilbert space $\mathscr{H}$, this is a natural constraint.

The study of bounded operators on Hilbert space has often been motivated by linear algebra. There is a popular, but naive, notion that finite dimensions are well understood. Operator theorists are keen to find the infinite-dimensional analogues of the finite-dimensional results. But they obtain even more pleasure when they find out why such analogues cannot hold.

Halmos, in his role as the master popularizer of this subject, has asked many questions about approximation of operators. In particular, his famous *Ten problems in Hilbert space* **[Ha2]** has provoked some of the most important work in this area. (See **[Ha3]** for the progress report.) Here we will mention two examples for motivation.

Everyone knows that a normal matrix can be diagonalized. This is not the case in infinite dimensions. For example, on $L^2(0,1)$, the operator given by $(Mf)(x) = xf(x)$ has no eigenvalues. Halmos asked: Is every normal operator the sum of a diagonal operator and a compact one? The answer (see §2) has had many ramifications.

On a finite-dimensional space, the set of nilpotent matrices is closed, and consists of all matrices with spectrum $\{0\}$. In infinite dimensions, the operators with spectrum $\{0\}$ (called *quasinilpotents*) are the operators satisfying

$$\lim_{n\to\infty} \|T^n\|^{1/n} = 0.$$

These need not be nilpotent, yet still may be limits of nilpotents. (For example, if N_k is the $k \times k$ nilpotent, Jordan block, let $T = N_1 \oplus \frac{1}{2}N_2 \oplus \frac{1}{3}N_3 \oplus \cdots$.) Halmos asked: Is every quasinilpotent operator the limit of nilpotents? The answer involves a deep analysis of the structure of arbitrary operators (see §4).

2. Normal operators. The appropriate analogue of diagonalization of matrices is the spectral theorem of Hilbert and von Neumann [**DS**]. This represents a normal operator as an integral over the spectrum of a certain "projection-valued measure". From this theorem, it is a simple exercise to show that a normal operator is the limit of diagonalizable ones. But for some purposes, this is not sufficient. In 1909, H. Weyl [**W**] showed that every Hermitian operator is the sum of a diagonal operator and a compact operator K of (arbitrarily) small norm.

Halmos's question was answered by David Berg [**B1**] and independently by W. Sikonia [**S**]. They showed:

THEOREM WNBS. *Let N be a normal operator on a separable Hilbert space. Given $\varepsilon > 0$, there is a diagonal operator D and a compact operator K with $\|K\| < \varepsilon$ such that $N = D + K$.*

This theorem is the cornerstone of the work of Brown, Douglas and Fillmore [**BDF1**]. They studied operators T such that $TT^* - T^*T$ is compact (*essentially normal operators*). These are precisely the operators such that the image $\pi(T)$ is normal in the quotient C^* algebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ by the compact operators. The original question which motivated their work was: When is an essentially normal operator equal to the sum of a normal operator and a compact one?

An operator T is *Fredholm* if $\pi(T)$ is invertible. The index group $\mathscr{A}^{-1}/\mathscr{A}_0^{-1}$ (where $\mathscr{A}_0^{-1}$ is the connected component of the identity in the group $\mathscr{A}^{-1}$ of invertible elements in $\mathscr{A}$) is isomorphic to the integers. Moreover, the canonical image of T in $\mathbf{Z}$ is called the *Fredholm index* and is given by the formula

$$\operatorname{ind} T = \operatorname{null}(T) - \operatorname{null}(T^*)$$

where $\operatorname{null}(T)$ is the dimension of the kernel (here necessarily finite).

Since $\|Nx\| = \|N^*x\|$ for every vector x and normal operator N, it follows that $\operatorname{ind}(N - \lambda) = 0$ whenever $N - \lambda$ is Fredholm. Index is invariant under compact perturbations, so if T is the sum of a normal operator and a compact one, then $\operatorname{ind}(T - \lambda) = 0$ whenever this is defined. This makes it easy to give a simple example of an essentially normal operator which is not of this form. Let S be the unilateral shift (i.e., $\{e_n\}_{n \geqslant 0}$ is an orthonormal basis and $Se_n = e_{n+1}$). An easy computation shows that $S^*S - SS^*$ is a rank one projection onto the span of $\{e_0\}$, so S is essentially normal. S has kernel $\{0\}$, and $\ker(S^*) = \operatorname{span}\{e_0\}$. Thus $\operatorname{ind} S = -1$, and S is not normal plus compact.

The converse is the remarkable result of Brown, Douglas, and Fillmore [**BDF1**]. The essential spectrum $\sigma(\pi T)$ of an operator T will be denoted $\sigma_e(T)$.

THEOREM BDF. *Suppose T is essentially normal and* $\operatorname{ind}(T - \lambda) = 0$ *for all λ in $\mathbf{C} \setminus \sigma_e(T)$. Then there is a normal operator N and a compact operator K such that $T = N + K$.*

To attack the problem, they considered the set of essentially normal operators T with fixed essential spectrum $X = \sigma_e(T)$. This set, modulo a natural equivalence relation, turns out to be a group $\mathrm{Ext}(X)$ under the group operation of direct sum. The WNBS Theorem can be interpreted as saying that $\mathrm{Ext}(X)$ has a unique trivial element. The proof of the BDF Theorem involves ideas from algebraic topology. They identify certain natural pairings with K-theory **[BDF2]**, and this has led to a tremendous amount of work on C^* algebras.

We only mention one more development in this direction—an important theorem of Dan Voiculescu **[V1]**. In the context of C^* algebras, it provides a trivial element for $\mathrm{Ext}(X)$ in many noncommutative situations. Here we are interested in a useful operator-theoretic corollary. If B is an operator, let $B^{(\infty)}$ denote $B \oplus B \oplus B \oplus \cdots$ acting on the l^2-direct sum of countably many Hilbert spaces.

THEOREM V. *Let T be a bounded operator. Let ρ be any separable representation of $C^*(\pi(T))$, and let $B = \rho(\pi(T))$. Given $\varepsilon > 0$, there is a compact operator K with $\|K\| < \varepsilon$ such that $T - K$ is unitarily equivalent to $T \oplus B^{(\infty)}$.*

This shows, among other things, that the set of operators with lots of reducing subspaces are dense in $\mathscr{B}(\mathscr{H})$. This answers another question of Halmos. And, this is a phenomenon quite distinct from finite dimensions, where the set of reducible matrices is a proper closed subset.

3. Quasitriangularity. One cannot discuss operator theory without briefly mentioning the invariant subspace probelm. Every matrix can be put into "triangular form", but the quest for an infinite-dimensional analogue has been elusive. A lot of important work has been done in this area in the past fifteen years, but we will not discuss it here. However, it led to the notion of a *quasitriangular* operator, again due to Halmos **[Ha1]**. An operator A is quasitriangular if there is an increasing sequence P_n of finite rank projections converging pointwise to the identity such that

$$\lim_{n \to \infty} \|(I - P_n)AP_n\| = 0.$$

Now $(I - P_n)AP_n = 0$ is equivalent to $P_n\mathscr{H}$ being invariant under A; so quasitriangular operators have a lot of "almost invariant" subspaces. This notion, but not the name, played a key role in Aronszajn and Smith's proof **[AS]** that every compact operator has an invariant subspace.

It is not difficult to show that A is quasitriangular if and only if $A = T + K$, where T is triangular with respect to some orthonormal basis, and K is compact. Such operators are limits of triangular operators because the norm of K can be made arbitrarily small. Normal operators and quasinilpotent operators are quasitriangular. The class is closed under compact perturbations and similarity transformations. It is norm closed and closed under taking polynomials and direct sums. Naturally upper triangular operators such as the backwards shift S^* are also quasitriangular.

It is perhaps more useful to know a nonquasitriangular operator. Such an operator is the unilateral shift S. Suppose $\mathscr{M}$ is a finite-dimensional subspace containing e_0. Since S is an isometry with range orthogonal to e_0, a dimension argument shows that some unit vector x in $\mathcal{M}$ is mapped to a vector Sx orthogonal to $\mathcal{M}$. An approximation argument now shows that

$$\lim_{n \to \infty} \|(I - P_n)SP_n\| = 1$$

for any sequence P_n of finite rank projections tending to the identity pointwise. Intuitively speaking, there was not enough room in $\mathcal{M}$ for $S\mathcal{M}$ because $\ker S$ is smaller than $\ker S^* = (\operatorname{Ran} S)^\perp$.

This generalizes in the following way. An operator T is called *semi-Fredholm* if it has closed range and one of null(T) or null(T^*) is finite. Index is extended to include $\pm\infty$ in the natural way. One obtains as above that if $T - \lambda$ is semi-Fredholm and $\operatorname{ind}(T - \lambda) < 0$, then T is not quasitriangular. Constantin Apostol, Ciprian Foiaş, and Dan Voiculescu prove the remarkable converse [**AFV1**].

THEOREM AFV1. *An operator T is quasitriangular if and only if* $\operatorname{ind}(T - \lambda) \geqslant 0$ *whenever $T - \lambda$ is semi-Fredholm.*

One striking corollary is that nonquasitriangular operators (rather than quasitriangular operators) have a ready supply of invariant subspaces. This is because, for such operators, there is a scalar λ so that $\operatorname{Ran}(T - \lambda)$ is a proper closed subspace. This subspace is invariant, not only for T, but for every operator commuting with T! So people searching for the elusive invariant subspace need only consider operators T such that both T and T^* are quasitriangular, the so-called *biquasitriangular* operators.

4. Nilpotents. Shortly after the results of the previous section were obtained, the same trio of Apostol, Foiaş, and Voiculescu cracked the Halmos problem mentioned in the introduction by giving a complete spectral characterization of the closure of the set of nilpotents. Several big steps had been taken earlier by Apostol [**A1, A2, AV**], Herrero [**H1**] and Voiculescu [**V2**].

Halmos was quick to point out in [**Ha2**] that his question was not quite right. An example of Kakutani shows that the limit of nilpotents can have a large spectrum. This phenomenon can occur because spectrum is not continuous, but only upper semicontinuous. Nonetheless, the semicontinuity of the spectrum implies that if T is the limit of nilpotents, then the spectrum $\sigma(T)$ is connected and contains 0. The same goes for the essential spectrum (by considering $\pi(T)$). Now for any $\lambda \neq 0$, and every nilpotent Q, $Q - \lambda$ is invertible and hence $\operatorname{ind}(Q - \lambda) = 0$. Since the set of semi-Fredholm operators is open and index is locally constant, one deduces that if T is the limit of nilpotents, then $\operatorname{ind}(T - \lambda) = 0$ wherever this index is defined. By Theorem AFV1, this is equivalent to saying that T is biquasitriangular. The second marvelous theorem of Apostol, Foiaş and Voiculescu [**AFV2**] can now be stated.

THEOREM AFV2. *An operator T is the limit of nilpotent operators if and only if*
(i) $\sigma(T)$ *is connected and contains* $\{0\}$,
(ii) $\sigma_e(T)$ *is connected and contains* $\{0\}$,
(iii) $\operatorname{ind}(T - \lambda) = 0$ *whenever $T - \lambda$ is semi-Fredholm.*

An important step in this argument is a theorem of Herrero [**H1**] which shows that if N is normal with connected spectrum including $\{0\}$, then N is the limit of nilpotents. There is now a fairly easy proof of this using a method of Berg [**B2**]. A short proof that every quasinilpotent operator is the limit of nilpotents is available in [**AFP**] based on Theorem V.

An operator T is said to be *algebraic* if it satisfies a polynomial identity $p(T) = 0$. Such operators are (nonorthogonal) algebraic direct sums of operators $\lambda_i + N_i$, where the N_i are nilpotent. Apostol and Foiaş [**AF**] and Foiaş, Pearcy, Voiculescu [**FPV**] develop canonical forms for biquasitriangular operators. This is improved upon by Voiculescu [**V2**], who uses it to prove

THEOREM V2. *The closure of the set of algebraic operators is the set of biquasitriangular operators.*

An interesting and shorter proof of both Theorem V2 and Theorem AFV2 is given in the books under review (Chapter 5). Again, this proof relies on Voiculescu's Theorem V, as well as on Theorem BDF.

5. Similarity orbits. The previous sections provide several examples of spectral descriptions of closed, similarity-invariant sets. A natural question along these lines is to ask for a description of the closure of

$$\mathscr{S}(T) = \{WTW^{-1}\colon W \text{ invertible}\},$$

the similarity orbit of a single operator T. Certain special cases follow using the techniques of the nilpotent case. For example, Barria and Herrero [**BH1**] show that if N is normal with perfect spectrum, then $\mathscr{S}(N)^-$ consists of all biquasitriangular operators A with perfect spectrum such that $\sigma_e(A)$ contains $\sigma(N)$ and every component meets $\sigma(N)$.

Apostol [**A3**] and Herrero [**H2**] independently show that if Q is quasinilpotent and Q^n is never compact, then Q is universal in the sense that $\mathscr{S}(Q)^-$ is the closure of all nilpotents. The general case, however, has required new techniques and analysis. The case of nilpotent operators is especially elusive and is not yet completely understood. But other than certain technicalities involving isolated points of the essential spectrum, Apostol, Herrero and Voiculescu have solved this problem. (See [**AHV**] for an announcement.)

The bulk of Chapters 7, 8 and 9 of the books under review is devoted to the proof of their theorem. Most of this material has not been published elsewhere. These results, being hot off the press, have not yet received widespread use. Yet they should prove to be very important. In particular, the methods of structural analysis are likely to become powerful tools for the operator theorist.

Even the finite-dimensional case is interesting, although elementary. Barria and Herrero [**BH2**] prove

THEOREM BH. *Let T in $\mathscr{M}_n$ be a matrix with minimal polynomial p. Then $\mathscr{S}(T)^-$ equals*

$$\{Q \in \mathscr{M}_n\colon \operatorname{rank} q(Q) \leqslant \operatorname{rank} q(T) \text{ for all } q \text{ which divide } p\}.$$

This result is useful in dealing with isolated eigenvalues of finite multiplicity in the infinite-dimensional case.

To get a handle on the general situation, let us collect together that spectral information which comes easily. Suppose A belongs to $\mathscr{S}(T)^-$. Then $\sigma(A)$ contains $\sigma(T)$, and each component of $\sigma(A)$ meets $\sigma(T)$. The same goes for $\sigma_e(A)$ relative to $\sigma_e(T)$. In fact, each component of

$$\sigma_{lre}(A) = \{\lambda\colon A - \lambda \text{ is not semi-Fredholm}\}$$

must meet $\sigma_e(T)$. As the set of semi-Fredholm operators is open, $\sigma_{lre}(A)$ contains $\sigma_{lre}(T)$ and

$$\operatorname{ind}(A - \lambda) = \operatorname{ind}(T - \lambda) \quad \text{for all } \lambda \notin \sigma_{lre}(A).$$

It is necessary to introduce one more notion. If A is semi-Fredholm, the *minimal index* is defined as

$$\min \operatorname{ind}(A) = \min\{\operatorname{nul} A, \operatorname{nul} A^*\}.$$

An operator is called *smooth* if $\min \operatorname{ind}(T - \lambda) = 0$ for all $\lambda \notin \sigma_{lre}(T)$. A smooth operator will be called *very smooth* if $\sigma_e(T)$ is perfect as well.

Now a special case of the Apostol-Herrero-Voiculescu theorem can be stated.

THEOREM AHV. *Suppose T belongs to $\mathscr{B}(\mathscr{H})$ and is very smooth. Then A belongs to $\mathscr{S}(T)^-$ if and only if $\sigma_{lre}(A)$ contains $\sigma_{lre}(T)$, each component of $\sigma_{lre}(A)$ meets $\sigma_e(T)$, and* $\operatorname{ind}(A - \lambda) = \operatorname{ind}(T - \lambda)$ *for all $\lambda \notin \sigma_{lre}(A)$.*

All of these theorems have dealt with spectral invariants for similarity-invariant sets. Once one has these invariants in hand, it is natural to attempt to compute the distance from an arbitrary operator to such a set. This can indeed be done in many cases. The interested reader is referred to Chapter 12 of the books for a good survey of the present knowledge and comprehensive references.

6. The books. Up until this point, I have been talking about the mathematics and not the books. The books deal with this marvelous material, most of which appears in book form for the first time. Especially in the first volume, which deals with the older material, the authors have taken pains to provide a unified treatment and more direct proofs than appear in the literature. It is a valuable tool for the researcher in the field.

It is less well suited for a neophyte or a casual reader. In the authors' effort to be comprehensive, they often state results in a long and tedious fashion. It would help a lot if major results were isolated in a clear and simply stated form. Still, an ambitious student can learn a lot from these books.

As I mentioned earlier, the material in section 5 appears in print in these volumes for the first time. I believe this is unfortunate. The proofs are long and involved. This material is bound to be put in a more palatable form in a few years. As it stands, it requires a substantial commitment from any reader intending to fight through these chapters. This material deserves to be better known, but I expect that this book will not remain the preferred source.

Volume II contains much material that has not been discussed here. It has a nice selection of topics, which is to a large extent work of the authors. Most of this appears in print elsewhere.

These books have an excellent bibliography. Notes at the end of each chapter enable readers to track down the references easily. Also, the index of notation is very handy. These features and the selection of material make these books a valuable tool for the operator theorist. It is hard work to go through these volumes, but it is worth the effort.

REFERENCES

[A1] C. Apostol, *On the norm-closure of nilpotents*, Rev. Roumaine Math. Pures Appl. **19** (1974), 277–282.

[A2] ______, *On the norm-closure of nilpotents*. III, Rev. Roumaine Math. Pures Appl. **21** (1976), 143–153.

[A3] ______, *Universal quasinilpotent operators*, Rev. Roumaine Math. Pures Appl. **25** (1980), 135–138.

[AF] C. Apostol and C. Foiaş, *On the distance to bi-quasitriangular operators*, Rev. Roumaine Math. Pure Appl. **20** (1975), 261–265.

[AFP] C. Apostol, C. Foiaş and C. M. Pearcy, *That quasinilpotent operators are norm-limits of nilpotent operators revisited*, Proc. Amer. Math. Soc. **73** (1979), 61–64.

[AFV1] C. Apostol, C. Foiaş and D. Voiculescu, *Some results on quasitriangular operators*. II–VI, Rev. Roumaine Math. Pures Appl. **18** (1973), 159–181, 309–324, 487–514.

[AFV2] ______, *On the norm-closure of nilpotents*. II, Rev. Roumaine Math. Pures Appl. **19** (1974), 549–577.

[AHV] C. Apostol, D. A. Herrero and D. Voiculescu, *The closure of the similarity orbit of a Hilbert space operator*, (NS) Bull. Amer. Math. Soc. **6** (1982), 421–426.

[AV] C. Apostol and D. Voiculescu, *On a problem of Halmos*, Rev. Roumaine Math. Pures Appl. **19** (1974), 283–284.

[AS] N. Aronszajn and K. T. Smith, *Invariant subspaces of completely continuous operators*, Ann. of Math. (2) **60** (1954), 345–350.

[BH1] J. Barria and D. A. Herrero, *Closure of similarity orbits of Hilbert space operators*. IV. *Normal operators*, J. London Math. Soc. (2) **17** (1978), 525–536.

[BH2] ______, *Closure of similarity orbits of nilpotent operators*. I. *Finite rank operators*, J. Operator Theory **1** (1979), 177–185.

[B1] I. D. Berg, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971), 365–371.

[B2] ______, *On approximation of normal operators by weighted shifts*, Michigan Math. J. **21** (1974), 377–383.

[BDF1] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C^*-algebras, operators with compact self-commutator and K-homology*, Bull. Amer. Math. Soc. **79** (1973), 973–978.

[BDF2] ______, *Extensions of C^*-algebras and K-homology*, Ann. of Math. (2) **105** (1977), 265–324.

[DS] N. Dunford and J. T. Schwartz, *Linear operators*, Part II: *Spectral theory. Self-adjoint operators in Hilbert space*, Interscience, New York and London, 1963.

[FPV] C. Foiaş, C. M. Pearcy and D. Voiculescu, *The staircase representation of a biquasitriangular operator*, Michigan Math. J. **22** (1975), 343–352.

[Ha1] P. R. Halmos, *Quasitriangular operators*, Acta. Sci. Math. (Szeged) **29** (1968), 283–393.

[Ha2] ______, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.

[Ha3] ______, *Ten years in Hilbert space*, Integral Equations Operators Theory **2** (1979), 529–564.

[H1] D. A. Herrero, *Normal limits of nilpotent operators*, Indiana Univ. Math. J. **23** (1974), 1097–1108.

[H2] ______, *Universal quasinilpotent operators*, Acta Sci. Math. (Szeged) **38** (1976), 291–300.

[R] G.-C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472.

[S] W. Sikonia, *The von Neumann converse of Weyl's theorem*, Indiana Univ. Math. J. **21** (1971), 121–123.

[V1] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), 97–113.

[V2] ______, *Norm-limits of algebraic operators*, Rev. Roumaine Math. Pures Appl. **19** (1974), 371–378.

[W] H. Weyl, *Über beschrankte quadratische Formen deren Differenzvollstetig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

KENNETH R. DAVIDSON

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 1, July 1986

0273-0979/86 $1.00 + $.25 per page

The bidual of $C(X)$. I, by S. Kaplan, Mathematics Studies, vol. 101, North-Holland, Amsterdam, The Netherlands 1984, xvi + 424 pp., $57.75 US/Dfl. 150.00. ISBN 0-444-87631-6

A *Riesz space* is a (real) linear space E endowed with a partial ordering $\leqslant$ which is translation-invariant (i.e. $x \leqslant y \Rightarrow x + z \leqslant y + z$) and a lattice (i.e. $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all x and y), and such that $\alpha x \geqslant 0$ whenever $x \geqslant 0$ in E and $\alpha \geqslant 0$ in $\mathbf{R}$. Write $E^+ = \{x\colon x \geqslant 0\}$. A *Riesz norm* on E is a norm $\| \ \|$ such that $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$, where $|x| = x \vee (-x)$. A *Banach lattice* is a Riesz space with a Riesz norm under which it is complete.

From the beginnings of functional analysis it has been recognized that many of the most important normed spaces are endowed naturally with Riesz space structures. The interactions of the three aspects of a Banach lattice—its linear, metric and order structures—lead to a rich and delightful, if not particularly deep, tapestry of interwoven motifs. We can study these either in the general, setting up an abstract theory, or in the particular, concentrating on well-known spaces of special importance. The book under review takes the latter course, though fully committed, in language and spirit, to the wider theory of normed Riesz spaces.

An *M-space* is a Banach lattice E in which $\|x \vee y\| = \max(\|x\|, \|y\|)$ whenever $x, y \in E^+$; an *L-space* is a Banach lattice E in which $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E^+$. There are effective representation theorems for both classes. A Banach lattice is an M-space iff it is isomorphic, as normed Riesz space, to the space $C_0(X)$ of continuous real-valued functions vanishing at infinity on some locally compact Hausdorff space X; it is an L-space iff it is isomorphic to the space $L^1(X)$ of equivalence classes of integrable real-valued functions on some measure space X. Among the M-spaces we naturally wish to identify those corresponding to compact spaces X; these are precisely the M-spaces with a *unit* e such that, for any x, $\|x\| \leqslant 1$ iff $|x| \leqslant e$.

Corresponding to the rich internal structure of Riesz spaces is an appropriately elaborate theory of morphisms between them. If E and F are Riesz