

ON EXTENDING SOLUTIONS TO WAVE EQUATIONS ACROSS GLANCING BOUNDARIES

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Introduction. The purpose of this note is to announce some results on the following extension problem. On a C^∞ manifold M with boundary, if u is a given extendible distribution satisfying

$$(1) \quad Pu \in C^\infty(M),$$

under what conditions (on $P, \partial M$, and $u|_{\partial M}$) can u be extended across ∂M as a solution, that is, to a distribution $\tilde{u} \in D'(\tilde{M})$ such that $P\tilde{u} \in C^\infty(\tilde{M})$, for some open manifold \tilde{M} extending M across ∂M ? Here P is assumed to be a second-order differential operator on M with smooth coefficients, noncharacteristic with respect to ∂M , and with real principal symbol p having fiber-simple characteristics

$$(2) \quad d_{\text{fiber}}p \neq 0 \quad \text{on} \quad p^{-1}(0) \cap (T^*M \setminus 0)$$

(for example, the wave operator acting in the exterior of a smooth obstacle). After extending the coefficients of P smoothly across ∂M , we can view P as an operator on some open extension \tilde{M} of M .

The problem is easily solved in the two cases where no null bicharacteristics tangent to ∂T^*M are present. When ∂M is everywhere elliptic with respect to P , classical theory implies that the desired \tilde{u} can be found if and only if $u|_{\partial M} \in C^\infty(\partial M)$. When ∂M is everywhere hyperbolic, nothing has to be assumed about $u|_{\partial M} \in D'(\partial M)$, for the extension \tilde{u} can be produced simply by solving the Cauchy problem in a neighborhood of ∂M with Cauchy data given by u . Here we are interested in the two cases where null bicharacteristics tangent to ∂T^*M to first order are present, the diffractive and gliding cases. An example given in [8] shows that if the boundary is diffractive, even when $u|_{\partial M}$ is smooth, it may happen that no extension as a solution (in fact, no extension \tilde{u} such that $\rho \notin WF P\tilde{u}$ where $\rho \in \partial T^*M$ is a point of null bicharacteristic tangency) exists. Our main result (Theorem 2) implies that, in contrast to the diffractive case, near gliding points extensions as microlocal solutions always exist when $u|_{\partial M}$ is smooth. We construct such an extension after showing that, near a gliding point $\sigma \notin WF u|_{\partial M}$, any distribution u satisfying $Pu \in C^\infty(M)$ has the series expansion given in Theorem 1. The proof of Theorem 2 makes essential use of the recent unified treatment of the diffractive and gliding parametrices [5], in which the eikonal and transport equations are solved on both sides of the boundary. Full proofs will appear in [9].

We proceed to recall some terminology.

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Boundary points. Let $\iota: \partial M \rightarrow M$ be the inclusion inducing the projection $\iota^*: \partial T^*M \rightarrow T^*\partial M$. Then the elliptic, hyperbolic, and glancing regions are respectively

$$E = \{\sigma \in T^*\partial M \setminus 0: p \neq 0 \text{ on the line } \iota^{*-1}(\sigma)\}.$$

$$H = \{\sigma \in T^*\partial M \setminus 0: p \text{ has (two) simple zeros in } \iota^{*-1}(\sigma)\}.$$

$$G = \{\sigma \in T^*\partial M \setminus 0: p \text{ has a double zero, } \rho, \text{ in } \iota^{*-1}(\sigma)\}.$$

Letting x be a real C^∞ function vanishing simply on ∂M and positive in $\overset{\circ}{M}$ near ∂M , we write $G = G_d \cup G_g \cup G_0$ (a union of the diffractive, gliding, and higher-order tangency regions), where $\sigma \in G_d, G_g$, or G_0 depending on whether $\{p, \{p, x\}\}(\rho) > 0, < 0$, or $= 0$ respectively.

Gliding parametrices. Let $\sigma = \iota^*(\rho) \in G_g$. Choose coordinates $(x, y, \xi, \eta) \in T^*\overset{\circ}{\mathbf{R}}^{n+1}$ such that $\overset{\circ}{M} = \{x > 0\}$, $\rho = (0, 0, 0, \bar{\eta})$, and $H_p y_1(\rho) > 0$ (so y_1 will serve as our time variable). The forward and backward gliding parametrices at σ are maps $B_\pm: D'(\partial M) \rightarrow D'(M)$ such that for some small neighborhood $U \subset M$ of $\pi\sigma$ and some small conic neighborhood $\Lambda \subset T^*\partial M \setminus 0$ of σ , we have $PB_\pm g \in C^\infty(U)$ and $B_\pm g|_{\partial M} = g \bmod C^\infty(\partial M)$ for all g with $WFg \subset \Lambda$. Moreover, B_+ (resp. B_-) propagates singularities in the direction of increasing (resp. decreasing) $y_1 \cdot B_\pm$ (see [2, 5, or 7]) are constructed from Fourier-Airy integral operators

$$(3) \quad C_\pm F(x, y) = (2\pi)^{-n} \int_{\Gamma_\pm} e^{i\phi(x, y, \mu)} I(a, b, \zeta; x, y, \mu) \hat{F}(\mu) d\mu,$$

where

$$I = [a(x, y, \mu) \text{Ai}(\zeta(x, y, \mu)) + b(x, y, \mu) \text{Ai}'(\zeta(x, y, \mu))] / \text{Ai}(\zeta_0(\mu)).$$

Here $\mu = (z, \mu') = (z, \mu_2, \dots, \mu_n) \in \mathbf{C} \times \mathbf{R}^{n-1}$ and $\text{Im } z = \mp T$ ($T > 0$) on Γ_\pm respectively. Ai is the standard Airy function (see [6, p. 218]), an entire function whose zeros are all simple and negative. $\zeta_0(\mu) = \zeta(0, y, \mu) = z\mu_n^{-1/3}$, so $(\text{Ai}(\zeta_0))^{-1}$ makes sense on Γ_\pm . The phase functions ϕ, ζ and the symbols a, b are obtained by taking almost analytic extensions (see [3]) in the μ_1 variable of functions solving eikonal and transport equations on both sides of $x = 0$. The symbols a and b are supported in a small conic neighborhood $\nu \subset \{(x, y, \mu): \mu_n \geq C|\mu|\}$ of $(0, 0, \bar{\mu})$, where $\bar{\mu} = (0, \dots, 0, 1)$. Finally, we recall that ϕ and b also satisfy $\phi'_{x, y}(0, 0, \bar{\mu}) = (0, \bar{\eta})$ and $b|_{x=0} = 0$.

Operators B_\pm with the desired properties can now be obtained by setting $B_\pm = C_\pm J$, where $J: E'(\partial M) \rightarrow E'(\mathbf{R}^n)$ is a proper elliptic F.I.O. microlocally inverting the boundary operators $(C_\pm)|_{x=0}$.

Note that although ϕ, ζ, a , and b are defined on both sides of $x = 0$, the operators B_\pm are defined *only* in $x \geq 0$ because the Airy quotients in (3) blow up exponentially in $x < 0$.

Main results. We number the zeros r_k of $\text{Ai}(z)$ so that $0 > r_1 > r_2 > \dots \rightarrow -\infty$.

THEOREM 1. *Let P be a second-order differential operator on M non-characteristic with respect to ∂M , with real principal symbol p satisfying (2). If $\sigma \in G_g$ and $u \in D'(M)$ satisfies $Pu \in C^\infty(M)$ and $\sigma \notin WF u|_{\partial M}$, then $u = v_1 + v_2$, where $\sigma \notin WF_b v_2$ (WF_b is defined in [4]) and $v_1 = \sum_k u_k =$*

$$i(2\pi)^{1-n} \sum_k \int e^{i\phi(x,y,\tilde{\mu}_k)} \alpha_k \mu_n^{1/3} [(a \operatorname{Ai}(\zeta) + b \operatorname{Ai}'(\zeta))(x, y, \tilde{\mu}_k)] \hat{F}(\tilde{\mu}_k) d\mu_2 \cdots d\mu_n.$$

Here $\tilde{\mu}_k = (r_k \mu_n^{1/3}, \mu_2, \dots, \mu_n)$, α_k is the residue of $(\operatorname{Ai}(z))^{-1}$ at r_k , a, b, ϕ, ζ are as in (3), and $F \in E'(\mathbf{R}^n)$.

Since $b(0, y, \mu) = 0$ and $\zeta(0, y, \mu) = z \mu_n^{-1/3}$, each of the terms u_k satisfies $u_k|_{x=0} = 0$ as well as $Pu_k \in C^\infty(U)$ for some neighborhood $U \subset M$ of $\pi\sigma$. The factors $(\operatorname{Ai}(\zeta_0))^{-1}$ which forced us to consider only $x \geq 0$ when defining B_\pm have now disappeared, so the fact that ϕ, ζ, a , and $b(x, y, \tilde{\mu}_k)$ satisfy eikonal and transport equations in a two-sided neighborhood $\tilde{U} \subset M$ of $\pi\sigma$ can be put to use. We deduce that each u_k extends to a $\tilde{u}_k \in D'(\tilde{U})$ such that $P\tilde{u}_k \in C^\infty(\tilde{U})$. This suggests

THEOREM 2. *Let P and M be as in Theorem 1. If $\sigma \in G_g$ and if $u \in D'(M)$ satisfies $Pu \in C^\infty(M)$ and $\sigma \notin WF u|_{\partial M}$, then an extension \tilde{u} can be constructed such that $WF P\tilde{u} \cap \iota^{*-1}(\Gamma) = \emptyset$, for some conic neighborhood $\Gamma \subset T^*\partial M \setminus 0$ of σ .*

Sketch of the proofs. Using the fact that regularity propagates in the boundary near gliding points (see [1]), we first find a distribution $g \in E'(\partial M)$, supported in $y_1 < 0$, for which $u = v_1 + v_2$ where $v_1 = B_{+g} - B_{-g}$ and $\sigma \notin WF_b v_2$. Putting $F = Jg$ we have $(2\pi)^n(B_{+g} - B_{-g}) =$

$$(4) \quad \int_{\Gamma_+} e^{i\phi} I(a, b, \zeta) \hat{F}(z, \mu') dz d\mu' - \int_{\Gamma_-} e^{i\phi} I \hat{F} dz d\mu'.$$

We compute the integral with respect to z first (noting that the integrand has poles at $r_k \mu_n^{1/3}$, $k = 1, 2, \dots$) by taking the limit as $k \rightarrow \infty$ of integrals around closed rectangular contours in \mathbf{C} centered at the origin. In [5] it is shown that the zeros σ_k of $\operatorname{Ai}'(z)$ satisfy $0 > \sigma_1 > r_1 > \sigma_2 > r_2 \cdots \rightarrow -\infty$. So it is convenient to make the left vertical side of the k th rectangle pass through $\sigma_{k+1} \mu_n^{1/3}$. Estimates of the Airy quotients on the vertical segments show that the contributions to the contour integrals from those segments approach zero as $k \rightarrow \infty$. Thus Cauchy's integral formula with remainder yields that (4) equals

$$(5) \quad 2\pi i \sum_k \int e^{i\phi(x,y,\tilde{\mu}_k)} \alpha_k \mu_n^{1/3} [(a \operatorname{Ai}(\zeta) + b \operatorname{Ai}'(\zeta))(x, y, \tilde{\mu}_k)] \hat{F}(\tilde{\mu}_k) d\mu' \\ + \iint_W \partial_{\bar{z}}(e^{i\phi} I(a, b, \zeta) \hat{F}) d\bar{z} \wedge dz d\mu',$$

where $W = \{z \in \mathbf{C}: -T \leq \operatorname{Im} z \leq T\}$. Though the integrand in the second term has infinitely many poles in W , estimates of $(\operatorname{Ai}(\zeta_0))^{-1}$ near the negative

real axis and the fact that ϕ, ζ, a , and b are almost analytic in z imply that the second term is smooth in $x \geq 0$. This finishes the proof of Theorem 1.

As noted above, each u_k extends to a $\tilde{u}_k \in D'(\tilde{U})$ such that $P\tilde{u}_k \in C^\infty(\tilde{U})$. Let $\chi(y) \in C_0^\infty(\mathbf{R}^n)$. Then repeated integrations by parts with respect to the y variable yield, for all $N > 0$, the estimates $|\langle \tilde{u}_k(x, \cdot), \chi(\cdot) \rangle| \leq C_N |\alpha_k| |r_k|^{-N}$ with C_N independent of k . Similar estimates clearly hold for $\partial_x^\beta \tilde{u}_k$ as well. Since $r_k \sim -ck^{2/3}$ (see [5, Appendix A]) and since the residues α_k can be shown to satisfy $|\alpha_k| \leq C|r_k|^{-1/4}$, we may conclude that the sum $\sum_k \tilde{u}_k$ converges to a distribution $\tilde{v}_1 \in C^\infty(R: D'(\mathbf{R}^n))$ extending v_1 . Moreover, for all $N > 0$, $|P\tilde{u}_k| \leq C_N |r_k|^{-N}$ in \tilde{U} (by the properties of ϕ, ζ, a , and b), and similar estimates hold for $\partial_{x,y}^\beta (P\tilde{u}_k)$. Hence $P\tilde{v}_1 = \sum_k P\tilde{u}_k \in C^\infty(\tilde{U})$. To complete the proof of Theorem 2, we invoke a simple lemma [8, 1.6] which implies that since $\sigma \notin WF_b v_2$, an extension \tilde{v}_2 of v_2 can be constructed such that $WF \tilde{v}_2 \cap \iota^{*-1}(\Gamma) = \emptyset$ for some conic neighborhood $\Gamma \ni \sigma$. So $\tilde{u} = \tilde{v}_1 + \tilde{v}_2$ is the desired extension of u .

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