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Invariance theory, the heat equation, and the Atiyah-Singer index theorem, by Peter B. Gilkey, Mathematics Lecture Series, vol. 11, Publish or Perish, Inc., Wilmington, Delaware, 1984, viii + 349 pp., $40.00. ISBN 0-914098-20-9

The announcement of what became known as the "Atiyah-Singer Index Theorem" appeared in this Bulletin in 1963. The full proof came out in the Annals of Mathematics in a series of papers between 1968 and 1971, although *Seminar on Atiyah-Singer Index Theorem*, edited by R. Palais, was published in 1965 and contained discussions of the proof and the background needed to understand the theorem. This beginning was typical of the development of the subject. A tremendous amount of research has been generated, yet there has been a relative scarcity of sources for the uninitiated. The book under review is the first detailed exposition of the approach to the index theorem developed by its author and independently by V. K. Patodi in the early seventies. My main complaint about the book is that it is long overdue. All this is partly explained by the heavy demand posed by the subject on both the student and the expository writer. What is required is considerable breadth. Familiarity with analysis, algebraic topology, and Riemannian geometry is an absolute minimum for the student. The subject is in constant flux and has interacted with (this list is certainly incomplete and the ordering is random) number theory, algebraic geometry, mathematical physics, representation theory of Lie groups, probability, and Riemannian geometry.

Index theory is the study of global aspects of systems of linear elliptic partial differential equations. One considers an elliptic operator D between spaces of C^∞ sections of two hermitian vector bundles E and F on a compact Riemannian manifold M. The adjoint operator D^* is also elliptic and, because of the ellipticity, the spaces of solutions of the equations $Du = 0$ and $D^*u = 0$ are finite dimensional. The index of D, $\operatorname{ind}(D)$, is defined as

$$\operatorname{ind}(D) = \dim\ker D - \dim\ker D^*.$$

The celebrated index formula of Atiyah and Singer computes this index as an integral of a *locally* defined expression. More precisely, the integrand is a differential form manufactured as a polynomial in the coefficients of D and their derivatives. The most classical example is the operator $d + \delta$ between the spaces of even and odd differential forms on a Riemannian manifold and the index formula reduces to the Chern-Gauss-Bonnet theorem. Examples are absolutely essential in this subject. All of the existing proofs proceed by proving the index formula for special operators (usually twisted signature operators), and then showing, by the use of K-theory, that this forces the formula to hold in general.

The original proof of the Atiyah-Singer theorem used global topological methods to obtain the local formula for the index. From that time the search was on for a proof that would produce the formula in a "natural" way. This search is by no means over. New approaches are being developed even now. In their work on $d + \delta$ and the eigenvalues of the Laplacian on forms, McKean and Singer used the heat equation to obtain a local formula for the Euler characteristic whose integrand was a polynomial in the components of the Riemann curvature tensor and its covariant derivatives. They conjectured that a "fantastic cancellation" took place and that the integrand obtained by the heat equation method was the Chern-Gauss-Bonnet integrand. All one knew at that point was that the two integrands had the Euler characteristic of the manifold M as their integral. Independently and almost simultaneously Gilkey and Patodi showed that Singer and McKean were justified in their optimism and that a remarkable cancellation of derivatives of the curvature did indeed take place. Generalizations of their methods to the Riemann-Roch theorem and the Hirzebruch signature theorem followed rapidly. Patodi's approach was to trace the algebra carefully to show explicitly that the cancellation took place. Gilkey, on the other hand, showed that derivatives of the curvature can be eliminated from the integrands produced by the heat equation method on *a priori* grounds.

How does the heat equation yield a local formula for the index? Consider the operators DD^* and D^*D on $C^\infty(F)$ and $C^\infty(E)$, respectively. Both are elliptic, selfadjoint and nonnegative. It is easy to see that for every $\lambda > 0$ the multiplicity of λ as an eigenvalue of D^*D is equal to the multiplicity of λ as an eigenvalue of DD^*. It follows that

$$\begin{aligned}\operatorname{ind}(D) &= \dim\ker(D^*D) - \dim\ker(DD^*)\\ &= \sum_\lambda m(\lambda)\exp(-\lambda t) - \sum_\lambda m^*(\lambda)\exp(-\lambda t)\end{aligned}$$

for every value of $t > 0$, where $m(\lambda)$ and $m^*(\lambda)$ denote the multiplicities of λ as the eigenvalue of D^*D and DD^* respectively. On the other hand, the fundamental solutions $k(x, y, t)$ and $k^*(x, y, t)$ of the parabolic equations

$$(DD^* + \partial/\partial t)u = 0 \quad \text{and} \quad (D^*D + \partial/\partial t)u = 0$$

can be represented in terms of eigensections and eigenvalues of the operators DD^* and D^*D as follows. If φ_γ and ψ_λ denote the normalized eigensections

D^*D and DD^* respectively, then

$$k(x, y, t) = \sum_\lambda m(\lambda)\exp(-\lambda t)\varphi_\lambda(x) \otimes \varphi_\lambda(y),$$

$$k^*(x, y, t) = \sum_\lambda m^*(\lambda)\exp(-\lambda t)\psi_\lambda(x) \otimes \psi_\lambda(y).$$

Setting $x = y$, taking the trace, and integrating over M, one obtains the formula

$$\operatorname{ind}(D) = \int_M \left[\operatorname{tr}(k(x, x, t)) - \operatorname{tr}(k^*(x, x, t))\right] d\,\mathrm{vol}.$$

On the other hand, the pointwise traces $k(x, x, t)$ and $k^*(x, x, t)$ have the following asymptotic expansion for $t \to 0$.

$$\operatorname{tr} k(x, x, t) \sim \sum_{k \geqslant -d} a_k(x) t^{k/2m}$$

$$\operatorname{tr} k^*(x, x, t) \sim \sum_{k \geqslant -d} a_k^*(x) t^{k/2m},$$

where $d = \dim M$, m is the order of the operator D, and $a_k(x)$, $a_k^*(x)$ are given by universal rational formulae in the coefficients of the operator D and their derivatives. Thus finally

$$\operatorname{ind}(D) = \int_M \left(a_0(x) - a_0^*(x)\right) d\,\mathrm{vol}.$$

The formulae for $a_0(x)$ and $a_0^*(x)$ are very complicated. However, one can deduce from their form certain qualitative properties, e.g., their behaviour under the scaling of the operator D. At this point invariance theory is used to identify the integrand for geometrically defined operators.

Invariance theory, in this context, is the study of invariants, under the action of a Lie group G, in the full tensor algebra of a vector space V. V appears as the tangent space of the manifold M, possibly tensored with the fiber of a coefficient bundle, and G is the product of the structure group of the tangent bundle and the structure group of the coefficient bundle. The point is, of course, that the integrand in the local formula for the index is such an invariant.

We now review briefly the contents of the book. Chapter 1 contains the analysis. Pseudodifferential operators are used to construct the fundamental solution of the heat equation and to prove the local formula for the index. In Chapter 2 characteristic classes are reviewed from the point of view of differential forms, and an axiomatic characterization of the Euler class and the Pontriagin classes is given. A similar axiomatic characterization is proved for mixed characteristic classes of the tangent space and a coefficient bundle. The Chern-Gauss-Bonnet formula is proved as an application of the characterization of the Euler class. All these results are due to the author and constitute his main contribution to the subject. The proofs are combinatorial in nature, rather lengthy and complicated. Because the author approaches his results from the point of view of general elliptic operators, he avoids the use of tensor calculus for the most part. For me personally this is a mixed blessing.

Chapter 3 is the core of the book. Special cases of the index theorem are proved here using the scheme outlined above. First it is shown, using the heat equation method, that the integrand in the local index formula has appropriate functorial properties, i.e., is an invariant. Invariance theory of Chapter 2 then shows that it must be a characteristic class. Finally the class is pinned down by computing suitably chosen examples. In this way the author gives a proof of the twisted Hirzebruch signature formula and the Riemann-Roch theorem. The general form of the Atiyah-Singer index formula is then proved using K-theory.

Of course the book contains much more. Chapter 4 is devoted to additional topics beyond the simplest version of the index theorem (index for manifolds with boundary, Atiyah-Bott-Lefschetz fixed-point formula, η-invariant, spectral geometry). The background for these additional topics is developed in the first two chapters.

The book is an excellent introduction to this beautiful and difficult subject. It is relatively self-contained and collects material previously scattered in research literature. It is very suitable to be a textbook for a graduate course, yet it takes the reader up to an area of active research. Gilkey's book will be very valuable to practitioners as well as students.

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