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THE CLASSIFICATION OF MAPS OF SURFACES

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In this note we discuss the topology of maps of positive degree between closed orientable surfaces. Two maps $f, g: M \to N$ are said to be *equivalent* if there exist homeomorphisms $h: M \to M$ and $k: N \to N$ such that $k \circ f = g \circ h$ (or $k \circ f \simeq g \circ h$ in the homotopy category). If k is homotopic to id_N we say f and g are *strongly equivalent*. The notion of equivalence is analogous to a change of basis in domain and range in linear algebra.

Surface maps of special interest are branched coverings, i.e., $f: M \to N$ is a *branched covering* if there exists a finite set of points $B \subset N$ such that $f|M - f^{-1}(B)$ is a covering map. An arbitrary branched covering may be approximated by a *generic branched covering*, i.e., one in which each point of N has degree (f) or degree $(f) - 1$ preimages.

One of the first people to study branched coverings was Riemann, who proved in his thesis (1851) that Riemann surfaces occur as conformal branched coverings of S^2. In 1871 and 1873 the classical function theorists Lüroth and Clebsch succeeded in showing that generic branched coverings of S^2 are classified up to (strong) equivalence by their degree. The classification problem for general range N was reduced by Hurwitz in 1891 to the algebraic-combinatorial study of representations of $\pi_1(N - B)$ into Σ_d, the symmetric group on d letters where $d =$ degree of the branched covering.

In 1928 Reidmeister showed that there is a 1-1 correspondence betweeen subgroups of $\pi_1(N)$ and covering spaces of N. This allows a generic branched covering $\phi: M \to N$ to be factored uniquely as a primitive (surjective on π_1) generic branched covering $\tilde{\phi}: M \to \tilde{N}$ followed by an unbranched covering map $p: \tilde{N} \to N$ corresponding to the image of ϕ on π_1.

Primitive generic branched coverings were shown to be classified by their degree by Hamilton in 1966 for arbitrary N provided that $b \geq 2d$, where b is the number of branch points and d is the degree. This was improved by Berstein and Edmonds in 1979 and 1984 to $b > d/2$ and arbitrary N, or with no restriction on b to $N = S^1 \times S^1$. More importantly, Berstein and Edmonds stressed that primitive generic branched coverings should be classified up to equivalence by their degree and they conjectured a suggestive normal form.

Recently we have shown that primitive generic branched coverings are actually classified up to strong equivalence by their degree, and consequently we prove the following theorem.

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THEOREM 1. *Two generic branched coverings $\phi, \psi: M \to N$ of closed orientable surfaces are strongly equivalent if and only if* degree(ϕ) = degree(ψ) *and* $\phi_{\#}\pi_1(M) = \psi_{\#}\pi_1(M)$.

As a corollary we deduce the homotopy classification of surface maps.

COROLLARY 2. *If $f, g: M \to N$ have positive degree then f and g are strongly equivalent in the pointed homotopy category if and only if* degree(f) = degree(g) *and* $f_{\#}\pi_1(M) = g_{\#}\pi_1(M)$.

PROOF. Diagram 1 summarizes results of Nielsen 1927 (column 1) and Edmonds 1978 (column 2). The entry in each box is a map which necessarily exists in a given homotopy class of maps from M to N. (A pinch is a map which contracts a subsurface of M with connected boundary to a point.)

$\simeq$	injective on π_1	primitive
degree 1	homeomorphism	pinch
degree > 1	covering map	generic branched covering

DIAGRAM 1

Since f and g may be written as primitive maps followed by the same covering map, the corollary follows directly from Theorem 1. □

Since surfaces are $K(\pi, 1)$'s the last corollary gives a classification of homomorphisms of surface groups.

COROLLARY 3. *If $f, g: G \to H$ are homomorphisms of surface groups of equal topological degree greater than zero such that $f(G) = g(G) \subset H$ then there exists an isomorphism $h: G \to G$ such that $f = g \circ h$.*

IDEA OF THE PROOF OF THE THEOREM. The proof of the theorem starts with the idea introduced by Gabai in his proof of the simple loop conjecture (1985) of factoring a map $\phi: M \to N$ as a branched immersion $s: M \to N \times I$ followed by a projection $\pi: N \times I \to N$. In this way the branched covering is "identified" with the space $s(M) \subset N \times I$.

As Figure 1 shows, slight changes in s may result in quite different sets of double curves in $s(M)$. By applying various such topological manipulations to s we eventually arrive at a branched immersion whose double curves when projected onto N are as in Figure 2.

Figure 2 shows the normal form for a primitive generic branched covering of degree d. The domain may be visualized as an immersed subset of $N \times I$ by first embedding d parallel copies of N in $N \times I$, then cutting the i and $i + 1$ copies along each curve labelled i, and finally interchanging sheets and gluing. See Figure 3.

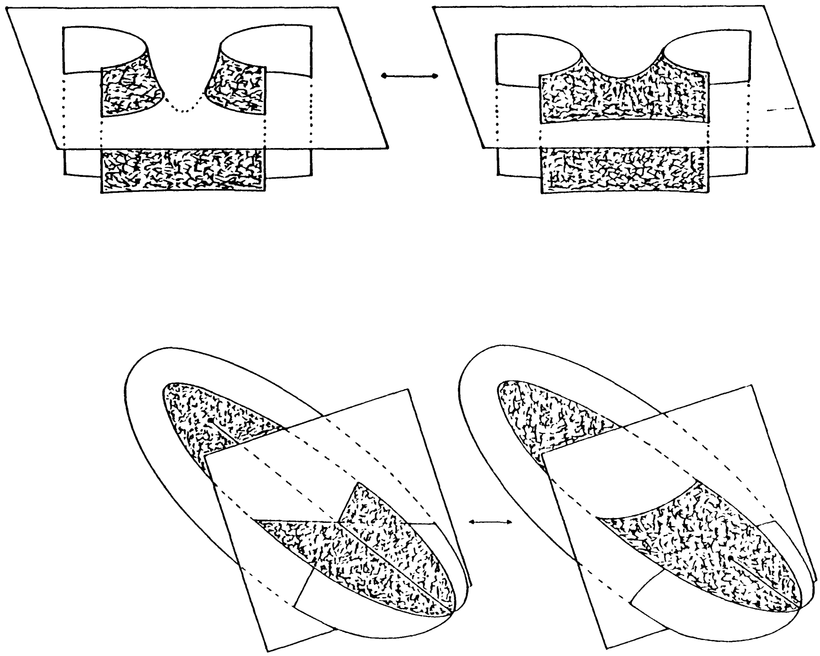

FIGURE 1

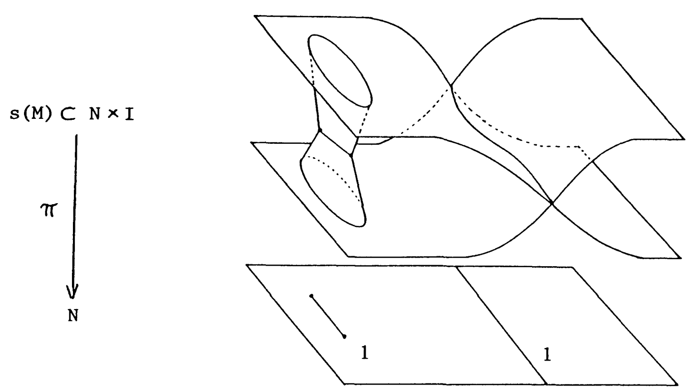

FIGURE 2

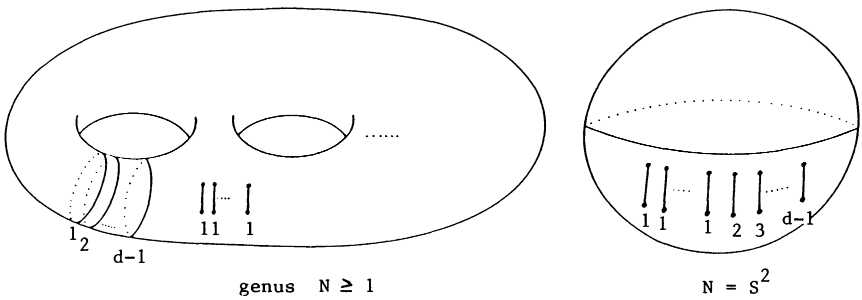

FIGURE 3

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