BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 1, July 1984

THE STRONG MULTIPLICITY ONE THEOREM FOR GL_n

BY CARLOS J. MORENO

Let k be a global field and denote by $\mathbf{A}$ its ring of adeles. Let ϕ be a unitary character of $\mathbf{A}^\times$ trivial on $k^\times$. For the group $G = \mathrm{GL}_n$, let $C_\phi^0 = L_2^0(G_\mathbf{A}/G_k, \phi)$ be the space of cusp forms on $G_\mathbf{A}$ with central character ϕ. The multiplicity one theorem, first proved by Jacquet-Langlands in [**2**] for $n = 2$, and by Shalika in [**11**] for all n, states that each irreducible unitary representation of $G_\mathbf{A}$ occurs with multiplicity at most one in C_ϕ^0. It is also well known that each irreducible unitary representation of $G_\mathbf{A}$ occurring in C_ϕ^0 has a factorization $\pi = \bigotimes_v \pi_v$, where each π_v is a representation of the local group $G_v = \mathrm{GL}_n(k_v)$. As a complement to the multiplicity one theorem, one has the following rigidity theorem (see [**9**, p. 209; **4**, p. 552]; for $n = 2$, [**1**, p. 307; **7**, p. 187]).

THEOREM (JACQUET–PIATETSKII-SHAPIRO–SHALIKA). *Suppose $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ are automorphic cuspidal irreducible representations of $\mathrm{GL}_n(k_\mathbf{A})$. Suppose S is a finite set of places, which in case $n > 2$ is assumed to contain only nonarchimedean places. Suppose $\pi_v \cong \pi'_v$ for all $v \notin S$. Then $\pi_v = \pi'_v$ for all v (hence $\pi \cong \pi'$).*

REMARK. In contradistinction to the multiplicity one theorem, the above result can be thought of as saying that given local representations π_v for all except a finite number of places v of k, there is at most one irreducible component π in the space of cusp forms with the given local factors.

In this note we announce an even stronger result; namely if two automorphic representations have equivalent local components at a suitable finite set of places, then they agree. Here we consider only the number field case, the function field case being somewhat simpler.

Let d_k be the discriminant of the number field k. If π_v is the local component of an automorphic representation π of $G_\mathbf{A}$ at an archimedean place v of k, then the set of complex numbers $\{\lambda_i(v)\}_{1\le i\le n}$ which appear in the definition of the local L-factor,

$$L_v(s, \pi_v) = \prod_{i=1}^{n} G_v(s - \lambda_i(v))$$

($G_\mathbf{R}(s) = \pi^{-s/2}\Gamma(s/2)$, $G_\mathbf{C}(s) = 2(2\pi)^{-s}\Gamma(s)$), is called the infinity type of π at v. If λ is a real number such that $\max_{v,i}(|\lambda_i(v)|) \le \lambda$, then we say that the infinity type of π is bounded by λ. Let f_π denote the conductor of π, i.e. the positive integer which appears in the functional equation [**3**, p. 83]

$$L(s, \pi) = \epsilon(\pi)(f_\pi d_k^n)^{1/2-s} L(1 - s, \tilde{\pi}).$$

Received by the editors April 19, 1983 and, in revised form, January 1, 1984.
1980 *Mathematics Subject Classification*. Primary 10D40, 12B40.

We can now state the new result.

MAIN THEOREM. *Let $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ be two cuspidal automorphic representations of $G_{\mathbf{A}} = \mathrm{GL}_n(\mathbf{A})$ whose conductors satisfy $\max(f_\pi, f_{\pi'}) \leq f$ and whose infinity types are bounded in absolute value by λ. Then there exist constants A and c which depend on λ and the field k such that the following is true: if $\pi_v \cong \pi'_v$ for all places v of k with norm $N(v) \leq \exp Af^c$, then $\pi_v \cong \pi'_v$ for all v and, hence, $\pi \cong \pi'$.*

REMARK. For $n = 2$ we can replace $\exp Af^c$ by Af^c. The proof also yields the fact that only class one representations need to be compared.

For $n \geq 3$ the inductive argument of Piatetskii-Shapiro and Shalika for the proof of their result quoted above is elegant and elementary; it depends on a lemma of Gelfand and Kazhdan [**9**, p. 211] which ascertains the equality of two Whittaker functions on $\mathrm{GL}_n(k_v)$, v nonarchimedean, when restricted to the Levi component of a maximal parabolic of GL_n. Our proof, on the other hand, is less elementary; it is based on the important work of Jacquet, Piatetskii-Shapiro and Shalika [**4**] concerning the analytic properties of L-functions of Rankin-Selberg type $L(s, \pi \times \pi')$. The new ingredients are of three types: (i) a generalization of the explicit formulas developed in [**8**] but now applied to the quotient of Rankin-Selberg L-functions, $L(s, \pi \times \tilde{\pi})/L(s, \pi' \times \tilde{\pi})$; (ii) a new proof of Shahidi's Theorem $L(1 + it, \pi \times \pi') \neq 0$ [**10**, p. 462], which actually yields zero free regions near the real axis; (iii) a study of the Deuring-Heilbronn phenomenon for $L(s, \pi \times \pi')$, which implies that the presence of an exceptional zero near $s = 1$ forces all other zeros to be relatively far away from $s = 1$. The rest of the proof mimics that of Linnik's theorem on the magnitude of the smallest prime in an arithmetic progression.

As a corollary of the method of proof we also obtain the following stronger version of a result of Jacquet and Shalika. If π_v is a class one factor of a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A})$, then, as in [**6**, p. 778], we denote by $\chi_\pi(v) = \mathrm{Tr}(B_v)$ the trace of the Langlands' class, i.e. B_v is the semisimple conjugacy class associated to π_v via the Satake isomorphism.

COROLLARY (JACQUET-SHALIKA). *With notation as above, let π_i be a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(A)$, $1 \leq i \leq g$. Suppose the π_i have their conductors bounded by f and their infinity types bounded by λ. Then there are constants A and c which depend only on λ and the field k such that, if for complex numbers $x_1, \ldots, x_g$ the relation*

$$(*) \qquad \sum_{i=1}^{g} x_i \chi_{\pi_i}(v) = 0$$

holds for all places v of k with norms $f < N(v) \leq \exp Af^c$, then $x_1 = \cdots = x_g = 0$.

REMARK. In [**6**, p. 807], Jacquet and Shalika obtain the conclusion in the corollary from the assumption that the linear relation $(*)$ holds for all places v of k except a finite number.

Some interesting applications of these ideas to the classification theory of automorphic representations of GL_n are given in [**5**, **6**]. The proof of the main theorem will be published elsewhere.

BIBLIOGRAPHY

1. W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314.

2. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL_2, Lecture Notes in Math., vol. 114, Springer-Verlag, 1970.

3. H. Jacquet, *Principal L-functions of the linear group*, Proc. Sympos. Pure Math., vol. 33, Part II, Amer. Math. Soc., Providence, R. I., 1979, pp. 63–86.

4. H. Jacquet, I. Piatetskii-Shapiro and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367–464.

5. H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations*. I, Amer. J. Math. **103** (1981), 499–558.

6. ——, *On Euler products and the classification theory of automorphic forms*. II, Amer. J. Math. **103** (1981), 777–815.

7. T. Miyake, *On automorphic forms on* GL_2 *and Hecke operators*, Ann. of Math. (2) **94** (1971), 174–189.

8. C. J. Moreno, *Explicit formulas in the theory of automorphic forms*, Lecture Notes in Math., vol. 626, Springer- Verlag, 1977, pp. 73–216.

9. I. Piatetskii-Shapiro, *Multiplicity one theorems*, Proc. Sympos. Pure Math., vol. 33, Part I, Providence, R. I., 1979, pp. 209–212.

10. F. Shahidi, *On nonvanishing of L-functions*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 462–464.

11. J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. (2) **100** (1974), 171–193.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801