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EVANS M. HARRELL II

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Lectures from Markov processes to Brownian motion, by Kai Lai Chung, A Series of Comprehensive Studies in Mathematics, Vol. 249, Springer-Verlag, New York, 1982, viii + 239 pp., $34.00. ISBN 0-3879-0618-5

What ultimately constitutes a good mathematics book? It seems to the reviewer that this is a function $f(e, r, c)$ of the variables e = effort needed to comprehend the book, r = reward in the form of valuable understanding gained, and c = cost of the book. Of these, the first two are highly dependent on the reader, and, given the first two, dependence on the third is completely individual, hence need not be discussed here. We assume f is decreasing in e and increasing in r. On these assumptions, Chung's book comes out very well indeed for the present reviewer. But let us beware. The reviewer has recently written a book [*Essentials of Brownian motion and diffusion*, Math. Surveys, vol. 18, Amer. Math. Soc., Providence, R.I., 1981] which complements Chung's book to a considerable degree. It gets one over the hard beginning (especially §1.3, Optional Times) without appreciably encroaching on the content. For the two books together one might suggest the title *From Brownian motion to Markov processes and back*.

At the other end of the scale, the reviewer has always had a hard time with Hunt's basic memoir on probabilistic potential theory, and with the classical text of Blumenthal and Getoor which followed it. Not only do they go too fast, they seem to go without color, emphasis, or motivation, in a way which makes extensive knowledge of classical potential theory practically a prerequisite. Not so with Chung's book. It may be described as a text on probabilistic potential theory in the disguise of a book on stochastic processes. The potential theory appears rather innocuously on page 80, and before the reader is aware of what is happening he is lured into potential theory for the rest of the book. This is not a derogatory assertion. The book has the effect of teaching a difficult subject without requiring the will to learn it.

Before discussing the subject matter, let us indicate some features of the presentation. The title may suggest a dissociated collection of lectures, but this is not the case. The book is an organized unity of successive topics, and the progression is from more basic to more applied in Chapters 1–3, and again in Chapter 4. Chapter 5 is built on all that precedes. The only hard connection to make is between Chapter 3 (Hunt processes) and Chapter 4 (Brownian motion). The distinction is no doubt made clear by §4.1 on spatial homogeneity, but the reviewer would have profited from a few more remarks on why Brownian motion is an especially favored case of a Hunt process. Which special properties are derived from spatial homogeneity, and which from path continuity? Of course, it is the spatial homogeneity which makes spheres play a special role in the definition of superharmonic function, and which gives the reversibility in time needed to show that semipolar sets are polar, but it is the path continuity which makes possible the localization of superharmonicity to a domain D, and the probabilistic solution of the Laplace and (real) Schrödinger equations in D.

Within this rigid framework, the book indeed proceeds as a series of lectures. Each topic is treated as an entity and gives its own special emphasis. The work is permeated with sage remarks and helpful hints to the reader. Happening to have read it around the holiday season, the reviewer can liken it to a large fruitcake — full of exotic and unexpected spices and delicacies. (What exactly, for instance, is a "true leger-demain due to E. B. Dynkin" in the form of a mathematics theorem? See p. 106, Exercise 5.) But if it is a fruitcake, it is also woven through with strands of steel where logic and rigour are concerned. For example, we did not encounter a single set of probability 0 that had escaped the author's attention.

The exercises are somewhat miscellaneous until one takes into account the opening remark (p. viii) that "correction of any inaccurate statement should be regarded as part of the exercise." No doubt the presence of a few misprints is unavoidable, but there also seems to be a tendency to ignore some basic sources (Ito and McKean, in particular, which contains a good deal of Chapter 5 in the Brownian case), while attributing false results to others (for instance, the "complement to Theorem 6 due to P. A. Meyer", p. 105, which does not even hold for the Poisson process). The charming feature of the exercises is that, given comprehension of the text, they are either easily amended or reasonably short. Thus one can really use them to test comprehension, and in many cases (such as the exercises on Hypothesis L) to increase comprehension, without undue effort.

Turning to the subject matter, this is not the place to embark on a lengthy critique of probabilistic potential theory. Indeed, it is debatable whether the present work, which stops just short of Hunt's balayage theorem and the introduction of additive (or multiplicative) functionals, is intended as a text on that subject. A fairer statement might be that it is a book on the use of Markov processes as a tool in classical real analysis, both in potential theory (Chapters 3 and 5) and in partial differential equations (Chapter 4). Nevertheless, a few remarks on that score may serve to orient the nonprobabilist.

A prime requisite for using Markov processes as a tool for analysis is to consider not just one process, but a family thereof, including a process starting at each point x. Only then can a function $f(x)$ always be represented as the value at $t = 0$ of a process $f(X_t)$ with probabilities and expectations P^x and E^x. For Markov processes, this means exclusion of any "branch points". There is a semigroup $P_t f(x)$ such that (i) $P_0 f(x) = f(x)$, (ii) $P_t f(x) = E^x f(X_t)$, and (iii) $E^x(f(X_{t+s})|X_u,\ u \leqslant t) = P_s f(X_t)$, $0 \leqslant s, t$. At $t = 0$, (i) and (ii) imply $f(x) = E^x f(X_0)$, and hence $P^x\{X_0 = x\} = 1$. The semigroup property follows from the Markov property (iii) by applying E^x: $P_{t+s} f(x) = P_t P_s f(x)$. Thus the Markov property implies a certain kind of propagation in time.

A potential, on the other hand, exhibits a certain kind of propagation in space, as illustrated by the domination principle or the operation of balayage. A basic fact of probabilistic potential theory is that, given a suitable potential kernel $K(x, B)$, there corresponds a Markov family X_t such that $K(x, B) = E^x \int_0^\infty I_B(X_t)\,dt$. Then for any potential function $f(x) = \int g(y) K(x, dy)$, $0 \leqslant g$, the composition

$$f(X_t) = E^{X_t}\int_0^\infty g(X_s)\,ds = E^x\left(\int_0^\infty g(X_s)\,ds \,|\, X_u, u \leqslant t\right) - \int_0^t g(X_s)\,ds$$

is a supermartingale for every P^x. This makes it possible to use martingale theorems to obtain properties of such f. On the other hand, it is basically the strong Markov property of X_t which leads to the fact that the balayage of f on a set A is given by $E^x f(X_{T_A})$, $T_A = \inf\{t > 0\colon X_t \in A\}$, except for x in a semipolar subset of A.

Thus from the standpoint of martingale theory it is precisely the time-homogeneous strong Markov property of X_t which translates the supermartingale property of $f(X_t)$ into a super-averaging property of f, and conversely. This brings to bear on such f a method of great power and flexibility, but it also has its limitations. Instead of a field of force emanating from a set A, as in classical potential theory, we have a process starting at the point x where the potential is measured. The direction of action is thus reversed, which may partly explain why the probabilistic theory has no real counterpart for forces or the effects of moving charges. Not surprisingly, one discovers that in order to duplicate (and extend) many results of classical potential theory, some duality assumptions must be introduced.

This is accomplished in the highly original last chapter of Chung's book. Instead of repeating the now classical Hunt duality assumptions, a new set of hypotheses is introduced, but it is not followed dogmatically. Instead, the treatment centers on four basic "principles": equilibrium, maximum, polarity, and energy. Each is discussed under alternative hypotheses and from different points of view.

This is sound pedagogy. It does little good to proceed in such matters as if there were a single best set of extra assumptions. But it does not disguise the fact that doing potential theory probabilistically can lead to complications. A famous probabilist was once heard to say that studying Hunt-style potential theory is a good way to grow old before one's time, and there is no doubt a grain of truth in the remark. The present book, however, succeeds to a remarkable degree in rejuvenating the subject. In any case one cannot cease to marvel at the dexterity with which its author walks the highwire between probability and analysis.

FRANK B. KNIGHT

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Combinatorial integral geometry with applications to mathematical stereology, by R. V. Ambartzumian, John Wiley & Sons, Somerset, New Jersey, 1982, xvii + 221 pp., $45.00. ISBN 0-4712-7977-3

In 1890 J. J. Sylvester, still creative at age 76, published a paper [7] entitled *On a funicular solution of Buffon's "problem of the needle" in its most general form*. The work is headed by the phrase "quaintly made of cords" from *The Two Gentlemen of Verona*, and is replete with well-executed drawings resembling complicated block and tackle devices. Although Sylvester's ropes serve him better than those of Shakespeare's Valentine, the mathematics suffers from the vagueness ever present in early writings on measure and probability.

To get a rough idea of the type of problem considered by Sylvester, suppose that a number of needles of various lengths are welded into a fixed planar configuration and then are "tossed at random" onto a plane ruled by equally spaced parallel lines. Calculate the probability that some single line will cross all of the needles.

The present book, *Combinatorial integral geometry*, by R. V. Ambartzumian, places the work of Sylvester in a rigorous setting while broadly extending the key idea in his paper. The main body of this book is based entirely on topics taken from the author's works; the order of the chapters seems to be generally in the chronological order of appearance of the corresponding research. Before proceeding, let us introduce a bit of formalism, simple by hindsight today, but denied Sylvester.

In E^2 an oriented line is determined by a unit normal $\mathbf{u}$ and a signed distance S to the origin. Thus $S^1 \times R$, conveniently realized as the cylinder $r = 1$ in E^3, is established as a natural coordinate system for the oriented lines of E^2. The usual surface measure on the cylinder transfers as a motion