

AN ELEMENTARY INTRODUCTION TO THE LANGLANDS PROGRAM

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PREFACE

In a recent issue of the Notices of the American Mathematical Society (April 1983, p. 273), as part of a very brief summary of *Progress in Theoretical Mathematics* presented to the Office of Science and Technology of the President of the United States by a briefing panel from the National Academy of Sciences chaired by William Browder, the general mathematical reader will find the following paragraphs:

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“The unifying role of group symmetry in geometry, so penetratingly expounded by Felix Klein in his 1872 Erlanger Program, has led to a century of progress. A worthy successor to the Erlanger Program seems to be Langlands’ program to use infinite dimensional representations of Lie groups to illuminate number theory.

That the possible number fields of degree n are restricted in nature by the irreducible infinite dimensional representations of $GL(n)$ was the visionary conjecture of R. P. Langlands. His far-reaching conjectures present tantalizing problems whose solution will lead us to a better understanding of representation theory, number theory and algebraic geometry. Impressive progress has already been made, but very much more lies ahead.”

The purpose of this paper is to explain what the Langlands program is about—what new perspectives on number theory it affords, and what kinds of results it can be expected to prove.

To begin with, Langlands’ program is a synthesis of several important themes in classical number theory. It is also—and more significantly—a program for future research. This program emerged around 1967 in the form of a series of conjectures, and it has subsequently influenced recent research in number theory in much the same way the conjectures of A. Weil shaped the course of algebraic geometry since 1948.

At the heart of Langlands’ program is the general notion of an “automorphic representation” π and its L -function $L(s, \pi)$. These notions, both defined via group theory and the theory of harmonic analysis on so-called adèle groups, will of course be explained in this paper. The conjectures of Langlands just alluded to amount (roughly) to the assertion that the other zeta-functions arising in number theory are but special realizations of these $L(s, \pi)$.

Herein lies the agony as well as the ecstasy of Langlands’ program. To merely state the conjectures correctly requires much of the machinery of class field theory, the structure theory of algebraic groups, the representation theory of real and p -adic groups, and (at least) the language of algebraic geometry. In other words, though the promised rewards are great, the initiation process is forbidding.

Two excellent recent introductions to Langlands’ theory are [Bo and Art]. However, the first essentially assumes all the prerequisites just mentioned, while the second concentrates on links with Langlands’ earlier theory of Eisenstein series.

The idea of writing the present survey came to me from Professor Paul Halmos, and I am grateful to him for his encouragement. Although the finished product is not what he had in mind, my hope is that it will still make accessible to a wider audience the beauty and appeal of this subject; in particular, I shall be pleased if this paper serves as a suitable introduction to the surveys of Borel and Arthur.

One final remark: This paper is *not* addressed to the experts. Readers who wish to find additional information on such topics as the trace formula, θ -series, L -indistinguishability, zeta-functions of varieties, etc., are referred to the (annotated) bibliography appearing after Part IV. I am indebted to Martin Karel and Paul Sally for their help in seeing this paper through to its publication.

I. INTRODUCTION

In this article I shall describe Langlands' theory in terms of the classical works which anticipated, as well as motivated, it. Examples are the local-global methods used in solving polynomial equations in integers, especially "Hasse's principle" for quadratic forms; the use of classical automorphic forms and zeta-functions to study integers in algebraic number fields; and the use of groups and their representations to bridge the gap between analytic and algebraic problems. Thus, more than one half of this survey will be devoted to material which is quite well known, though perhaps never before presented purely as a vehicle for introducing Langlands' program.

To give some idea of the depth and breadth of Langlands' program, let me leisurely describe one particular conjecture of Langlands; the rest of this paper will be devoted to adding flesh (and pretty clothes) to this skeletal sketch (as well as defining all the terms alluded to in this Introduction!).

In algebraic number theory, a fundamental problem is to describe how an ordinary prime p factors into "primes" in the ring of "integers" of an arbitrary finite extension E of \mathbf{Q} . Recall that the ring of integers O_E consists of those x in E which satisfy a monic polynomial with coefficients in \mathbf{Z} . Though O_E need not have unique factorization in the classical sense, every *ideal* of O_E must factor uniquely into prime *ideals* (the "primes" of O_E). Thus, in particular,

$$(*) \quad pO_E = \prod \mathfrak{P}_i,$$

with each \mathfrak{P}_i a prime ideal of O_E , and the collection $\{\mathfrak{P}_i\}$ completely determined by p .

Now suppose, in addition, that E is *Galois* over \mathbf{Q} , with Galois group $G = \text{Gal}(E/\mathbf{Q})$. This means that E is the splitting field of some monic polynomial in $\mathbf{Q}[x]$, and G is the group of field automorphisms of E fixing \mathbf{Q} pointwise. According to a well-known theorem, each element of G moves around the primes \mathfrak{P}_i "dividing" p , and G acts transitively on this set. Thus the "splitting type" of p in O_E is completely determined by the size of the subgroup of G which fixes any \mathfrak{P}_i , i.e., by the size of the "isotropy groups" G_i (which are conjugate in G).

For simplicity, we shall now assume that the primes \mathfrak{P}_i in (*) are *distinct*, i.e., the prime p is *unramified* in E . In this case, the afore-mentioned isotropy groups are cyclic. To obtain information about the factorization of such p , attention is focused on the so-called *Frobenius element* $\text{Fr}_{\mathfrak{P}_i}$ of G , the canonical generator of the subgroup of G which maps any \mathfrak{P}_i into itself. (We shall discuss all these matters in more detail in II.C.2.) To be sure, $\text{Fr}_{\mathfrak{P}_i}$ is an automorphism of E over \mathbf{Q} determined *only up to conjugacy* in G . Nevertheless, the resulting *conjugacy class* $\{\text{Fr}_p\}$ completely determines the factorization type of (*). For example, when $\{\text{Fr}_p\}$ is the class of the identity alone, then (and only then) p *splits completely* in E , i.e., p factors into the maximum number of primes in O_E (namely $r = [E:\mathbf{Q}] = \#G$).

In general, one seeks to describe $\{\text{Fr}_p\}$ (and hence the factorization of p in E) intrinsically in terms of p and the arithmetic of \mathbf{Q} . To see what this means, consider the example

$$E = \mathbf{Q}(i) = \{\alpha + \beta i: \alpha, \beta \in \mathbf{Q}\},$$

with $O_E = \mathbf{Z}(i) = \{n + mi : n, m \in \mathbf{Z}\}$. In this case, $G = \text{Gal}(E/\mathbf{Q}) = \{I, \text{complex conjugation}\}$, and some elementary algebra shows that

$$\text{Fr}_p = \begin{cases} I & \text{if } -1 \text{ is a quadratic residue mod } p, \\ \text{conjugation} & \text{otherwise.} \end{cases}$$

For convenience, let us identify $\text{Gal}(E/\mathbf{Q})$ with the subgroup $\{\pm 1\}$ of \mathbf{C}^\times via the obvious isomorphism $\sigma: G \rightarrow \{\pm 1\}$. Then we have

$$\sigma(\text{Fr}_p) = (-1/p),$$

with $(-1/p)$ the Legendre symbol (equal to 1 or -1 according to whether -1 is, or is not, a quadratic residue mod p). To express this condition in terms of a congruence condition on p instead of on -1 , we appeal to a part of the quadratic reciprocity law for \mathbf{Q} which states that (for odd p , precisely those p unramified in $\mathbf{Q}(i)$)

$$(-1/p) = (-1)^{(p-1)/2}, \quad \text{i.e.,} \quad \sigma(\text{Fr}_p) = (-1)^{(p-1)/2}.$$

This is the type of intrinsic description of Fr_p we sought; from it, and the fact that

$$(-1)^{(p-1)/2} = 1 \Leftrightarrow p \equiv 1 \pmod{4},$$

we conclude that the factorization of p in $\mathbf{Z}(i)$ depends only on its residue modulo 4. In particular, all primes in a given arithmetic progression mod 4 have the same factorization type in $\mathbf{Z}(i)$. Moreover, since all the prime ideals of $\mathbf{Z}(i)$ are principal, and of the form (n) or $(n + im)$, we obtain the following:

THEOREM (FERMAT 1640, EULER 1754). *Suppose p is an odd prime. Then p can be written as the sum of two squares $n^2 + m^2$ if and only if $p \equiv 1 \pmod{4}$.*

PROOF. $p = n^2 + m^2 = (n + im)(n - im)$ if and only if p splits completely in $\mathbf{Z}(i)$.

A major goal of class field theory is to give a similar description of $\{\text{Fr}_p\}$ for arbitrary Galois extensions E . However, this goal is far from achieved and, in general, is probably impossible.

In general, we cannot expect there to be a modulus N such that $\{\text{Fr}_p\} = \{I\}$ if and only if p lies in some arithmetic progression mod N . However, if E is *abelian*, i.e., $G = \text{Gal}(E/\mathbf{Q})$ is abelian, then a great deal can be said. Indeed, suppose E is such an extension, and $\sigma: G \rightarrow \mathbf{C}^\times$ is a homomorphism. Then it is known that there exists an integer $N_\sigma > 0$ and a Dirichlet character

$$\chi_\sigma: (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{C}^\times \quad \text{such that} \quad \sigma(\text{Fr}_p) = \chi_\sigma(p)$$

for all primes p (unramified in E). This is E. Artin's famous and fundamental *reciprocity law of abelian class field theory*.² It implies—just as in the special case $E = \mathbf{Q}(i)$ —that the splitting properties of p in E depend only on its

²The more familiar form of this law directly identifies $\text{Gal}(E/\mathbf{Q})$ with the idele class group of \mathbf{Q} modulo the “norms from E ”; we stress the “dual form” of this assertion only because its formulation seems more amenable to generalization (i.e., nonabelian E).

residue modulo some *fixed modulus* N (depending on E). To see how this result directly generalizes the classical result of Fermat and Euler, we note that when $E = \mathbf{Q}(i)$ and $\sigma: G \rightarrow \{\pm 1\}$ is as before,

$$\sigma(\mathrm{Fr}_p) = \chi_\sigma(p),$$

with $\chi: (\mathbf{Z}/4\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$ defined as follows:

$$\chi_\sigma(n) = (-1)^{(n-1)/2}.$$

For more general abelian extensions, Artin's theorem not only implies the general quadratic reciprocity law (in place of the supplementary rule $(-1/p) = (-1)^{(p-1)/2}$) but also the so-called higher reciprocity laws of abelian class field theory. For a discussion of such matters, see, for example, [Goldstein, Tate, or Mazur].

The question remains: for *nonabelian* Galois extensions, how can the family $\{\mathrm{Fr}_p\}$ be described in terms of the ground field \mathbf{Q} ?

Recognizing the utility of studying groups in terms of their matrix representations, Artin focused attention on homomorphisms of the form $\sigma: \mathrm{Gal}(E/\mathbf{Q}) \rightarrow \mathrm{GL}_n(\mathbf{C})$, i.e., on n -dimensional *representations* of the Galois group G . In this way he was able to transfer the problem of analyzing certain conjugacy classes in G to an analogous problem inside $\mathrm{GL}_n(\mathbf{C})$ (where such classes as $\{\sigma(\mathrm{Fr}_p)\}$ are completely determined by their characteristic polynomials $\det[I_n - \sigma(\mathrm{Fr}_p)p^{-s}]$). By also introducing the (Artin) L -functions

$$L(s, \sigma) = \prod_p \left(\det[I_n - \sigma(\mathrm{Fr}_p)p^{-s}] \right)^{-1}$$

(whose exact definition will be given in II.C.2), Artin was further able to reduce this problem to one involving the *analytic* objects $L(s, \sigma)$.

Problem. Can the L -functions $L(s, \sigma)$ be defined in terms of the arithmetic of \mathbf{Q} alone?

It was in the context of this problem that Artin proved his fundamental reciprocity law. Indeed, for abelian E over \mathbf{Q} , and one-dimensional σ , Artin proved that his $L(s, \sigma)$ is identical to a Dirichlet L -series

$$L(s, \chi) = \prod (1 - \chi(p)p^{-s})^{-1}$$

for an appropriate choice of character $\chi: (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$.

For arbitrary E and σ , Artin was able to derive important analytic properties of $L(s, \sigma)$. However, what he was unable to do was discover the appropriate " n -dimensional" analogues of Dirichlet's characters and L -functions. Although some such 2-dimensional "automorphic" L -functions were being studied nearby (and concurrently) by Hecke, it remained for Langlands (40 years later) to see the connection and map out some general conjectures.

Roughly speaking, here is what Langlands did. He isolated the notion of an "automorphic representation of the group GL_n over the adèles of \mathbf{Q} " as the appropriate generalization of a Dirichlet character. Furthermore, he associated L -functions with these automorphic representations, generalizing Dirichlet's

L -functions in the case $n = 1$. Finally, he *conjectured* that each n -dimensional Artin L -function $L(s, \sigma)$ is exactly the L -function $L(s, \pi_\sigma)$ for an appropriate automorphic representation π_σ of GL_n . This is discussed—with an arbitrary number field F in place of \mathbf{Q} —in Part IV of the present paper; cf. Conjecture 1 in IV.A.

The (conjectured) correspondence $\sigma \rightarrow \pi_\sigma$ is to be regarded as a far reaching generalization of Artin's reciprocity map $\sigma \rightarrow \chi_\sigma$. In case $n = 2$, when π_σ corresponds to a classical automorphic form $f(z)$ in the sense of Hecke (see I.B), the map $\sigma \rightarrow \pi_\sigma$ affords an interpretation of the classes $\{\text{Fr}_p\}$ in terms of certain conjugacy classes in $GL_2(\mathbf{C})$ determined by the Fourier coefficients of the form $f(z)$. In general, the proper formulation of this conjecture (and other conjectures of Langlands) requires a synthesis and further development of all the themes alluded to heretofore: local-global principles, automorphic forms, group representations, etc.

In Part II of this paper, I motivate the use of p -adic numbers and adèles and survey Hecke's theory of automorphic forms, the L -functions of Artin and Hecke, and the use of group representations in number theory. Perforce, this brings us to the theory of *infinite-dimensional* representations of real and p -adic groups.

In Part III these "classical" themes and ingredients are mixed together to produce the all-important notion of an "automorphic representation of GL_n over \mathbf{Q} ". Finally, in Part IV, I survey the high points of Langlands' general program, with an emphasis on its historical perspective, and a brief description of techniques and known results.

II. CLASSICAL THEMES

A. The local-global principle. One of the major preoccupations of number theory in general has been finding integer solutions of polynomial equations of the form

$$(1) \quad P(x_1, x_2, \dots, x_n) = 0.$$

For convenience, let us assume that P is actually a homogeneous polynomial, and let us agree that only *nonzero* solutions are of interest. The difficulty in solving (1) is illustrated by Fermat's famous unproved assertion that the particular equation

$$X^n + Y^n - Z^n = 0$$

has no nontrivial solutions in integers for $n > 2$. Indeed, much of the development of the theory of algebraic numbers is linked to attempts by people contemporary with Kummer to solve this problem.

On the other hand, a question which is more easily decided is the existence of integral solutions "modulo m ". Clearly a *necessary* condition that integer solutions of (1) exist is that the congruence

$$(2) \quad P(x_1, \dots, x_n) \equiv 0 \pmod{m}$$

be solvable for *every* value of the modulus m . This observation leads naturally to the "local methods" we shall now explain.

Suppose $m = NM$ with N and M relatively prime. By the Chinese Remainder Theorem, (2) has a solution if and only if the similar congruences for N and M do. In other words, to solve (2) it is sufficient to solve congruences modulo p^k for any prime p and all positive integers k .

Whenever we focus on a fixed prime p , we say we are working “locally”. So suppose we fix a prime p and ask whether the congruence

$$(3) \quad P(x_1, \dots, x_n) \equiv 0 \pmod{p^k}$$

has a solution for all natural numbers k . It was Hensel who reformulated this question in a formal, yet significant, way in 1897. For each prime p he introduced a new field of numbers—the “ p -adic numbers”—and he showed that the solvability of (3) for all k is equivalent to the solvability of (1) in the p -adic numbers. Thus the solvability of the congruence (2) for all n is equivalent to the solvability of (1) in the p -adic numbers for all p .

Let us return now to the original problem of solving (1) in ordinary integers. In addition to being able to solve (2) modulo all integers m , it is also clearly necessary to be able to find real solutions for (1). The question of when these obviously necessary conditions are also sufficient is much more difficult, since the assertion that “an equation is solvable if and only if it is solvable modulo any integer and has real solutions” is in general false, or at least not known. For example, the Fermat equation has been known to be solvable p -adically for all p since around 1909.

On the other hand, there are important instances where this “local-global principle” is known to work.

THEOREM (HASSE-MINKOWSKI). *Suppose*

$$Q(x_1, \dots, x_n) = \sum_{i,j=1}^n a_{ij} X_i X_j$$

is a quadratic form with a_{ij} in \mathbf{Z} and $\det(a_{ij}) \neq 0$. Then $Q(x_1, \dots, x_n) = 0$ has a nontrivial integer solution if and only if it has a real solution and a p -adic solution for each p .

In order to give a more symmetric form to this example of the local-global principle, let me recall how the p -adic numbers can be constructed analogously to the real numbers. Fixing a prime p , we can express any fraction x in the form $p^\alpha n/m$, with n and m relatively prime to each other and to p . Then an absolute value is defined on \mathbf{Q} by

$$|x|_p = p^{-\alpha},$$

and the field of p -adic numbers is just the completion of \mathbf{Q} with respect to this metric $|\cdot|_p$. Note that the integer α (called the p -adic order of x) can be negative, and the integers that are close to zero “ p -adically” are precisely the ones that are highly divisible by p . Though perhaps jarring at first, this p -adic notion of size is entirely natural given our earlier motivations: the congruence $n \equiv 0 \pmod{p^k}$, with k large, translates into the statement that n is close to zero (p -adically).

Because \mathbf{R} is the completion of \mathbf{Q} with respect to the usual absolute value $|\cdot|$, it is customary to write $|x|_\infty$ for $|x|$, \mathbf{Q}_∞ for \mathbf{R} , and then call \mathbf{R} the completion of \mathbf{Q} at “the infinite prime” ∞ . The result is a family of locally compact complete topological fields \mathbf{Q}_p which contain \mathbf{Q} , one for each $p \leq \infty$. Each \mathbf{Q}_p is called a “local field”, and \mathbf{Q} itself is called a “global field”. With this terminology the Hasse-Minkowski theorem takes the following symmetric form: a quadratic form over \mathbf{Q} has a global solution if and only if it has a local solution for each prime p .

For the purposes of this article, the significance of the local-global principle is this: global problems should be analyzed purely locally, and with equal attention paid to each of the local “places” \mathbf{Q}_p .

Note. For a leisurely discussion of p -adic numbers, and instances of the local-global principle, the reader is urged to browse through the Introduction to [Bo Shaf and Cassels]. Also highly recommended is the expository article [Rob 2].

B. Hecke theory and the centrality of automorphic forms. In the 19th century the arithmetic significance of automorphic forms was clearly recognized, and examples of such forms were used to great effect in number theory.

Around 1830, Jacobi worked with the classical theta-function $\theta(z)$ in order to obtain exact formulas for the representation numbers of n as a sum of r squares. Then 30 years later, Riemann exploited this same function in order to derive the analytic continuation and functional equation of his famous zeta-function $\zeta(s)$.

Before explaining these matters in more detail, let us briefly recall the classical notion of an automorphic form.

1. *Basic notions.* Let H denote the upper half-plane in \mathbf{C} , and regard the group

$$\mathrm{SL}_2(\mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ real, } ad - bc = 1 \right\}$$

as the group of fractional linear transformations of H . An automorphic form of weight k is a function $f(z)$ which is holomorphic in H and “almost” invariant for the transformations $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in some *discrete* subgroup Γ of $\mathrm{SL}_2(\mathbf{R})$, i.e.,

$$(1) \quad f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Γ .

The most famous example of an automorphic form is the classical theta-function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z} = 1 + \sum_{n=1}^{\infty} 2e^{\pi i n^2 z}.$$

This is an automorphic form of weight $\frac{1}{2}$ for the group

$$\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \text{SL}_2(\mathbf{Z}): b, c \equiv 0 \pmod{2}, a, d \equiv 1 \pmod{2} \right\};$$

moreover,

$$\theta(-1/z) = (-iz)^{1/2} \theta(z).$$

More generally, let $Q_r(x_1, \dots, x_r)$ denote the quadratic form $\sum_{i=1}^r x_i^2$, and set

$$\theta_r(z) = \sum_{(n_1, \dots, n_r)} e^{\pi i Q_r(n_1, \dots, n_r)z},$$

the sum extending over all “integral” vectors (n_1, \dots, n_r) . Then $\theta_r(z)$ is again an automorphic form, this time of weight $r/2$. This example has special number theoretic significance because the coefficients in the Fourier expansion of this periodic function are the representation numbers of the quadratic form Q_r . Indeed, if $r(n, Q_r)$ denotes the number of distinct ways of expressing n as the sum of r squares, then

$$\theta_r(z)^r = \sum_{n=0}^{\infty} r(n, Q_r) e^{\pi i n z}.$$

Here are some more examples of automorphic forms:

(i) Let $\Delta(z)$ denote the function defined in H by

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}.$$

It is an automorphic form of weight 12 for the full modular group $\Gamma = \text{SL}_2(\mathbf{Z})$, and its Fourier coefficients $\tau(n)$ —carefully investigated by Ramanujan in 1916—are closely related to the classical partition function $p(n)$.

(ii) For $k > 1$ the function

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{\substack{(c,d) \neq (0,0) \\ \text{in } \mathbf{Z}^2}} \frac{1}{(cz + d)^{2k}}$$

is called the (normalized) *Eisenstein series* of weight $2k$. It is again an automorphic form with respect to the full modular group $\text{SL}_2(\mathbf{Z})$, this time with Fourier expansion

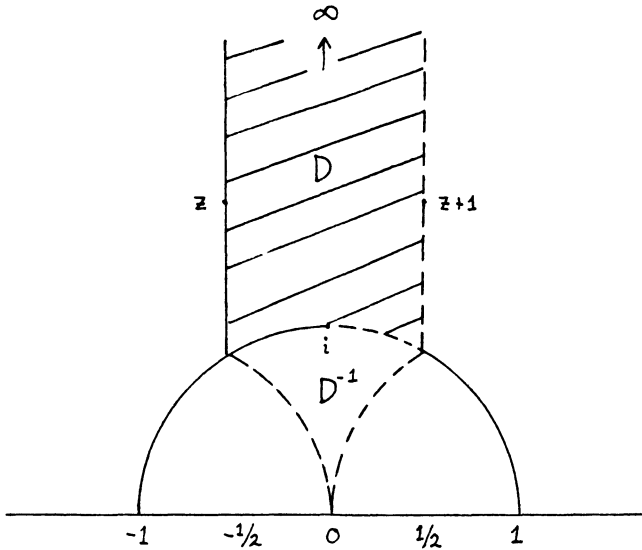
$$E_{2k}(z) = 1 + \frac{(-1)^k 4k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z},$$

with B_k the so-called n th Bernoulli number, and $\sigma_r(n) = \sum_{d|n} d^r$.

From these few examples, it is already clearly indicated that automorphic forms comprise an integral part of number theory. Indeed, invariance of the form with respect to translations of the type $z \rightarrow z + h$ implies the existence of a Fourier expansion $\sum a_n e^{2\pi i k z/h}$, with the a_n of number-theoretic significance.

In general, the automorphy property (1) implies $f(z)$ is determined by its values on a “fundamental domain” D for the action of Γ in H . More precisely,

D is a subset of H such that every orbit of Γ (with respect to the action $z \rightarrow (az + b)/(cz + d)$) has exactly one representative in D . For example, for $\Gamma = \text{SL}_2(\mathbf{Z})$, the fundamental domain D looks like this:



Note that any other fundamental domain must be obtained by applying to this D some $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Γ . In particular, the domain D^{-1} pictured above is precisely the image of D by the “inversion” element $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the point “at infinity” for D being mapped to the “cusp” at 0 in (the boundary of) the fundamental domain D^{-1} .

To be able to apply convenient methods of analysis to the study of automorphic forms, it is customary to impose additional technical restrictions on the regularity of f at “cusps” along the boundary of a fundamental domain, especially “at infinity”. This implies in particular that $f(z)$ always has a Fourier expansion of the form

$$(2) \quad f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z / h}.$$

For example, for $\Delta(z)$ or $E_{2k}(z)$ we can take $h = 1$, but for $\theta(z)$, which is an automorphic form only on $\Gamma(2)$ (which does not contain the translation $z \rightarrow z + 1$), the period is no longer 1, and we must take $h = 2$.

Let us denote by $M_k(\Gamma)$ the vector space of automorphic forms of weight k for Γ which are “regular at the cusps” of Γ , and by $S_k(\Gamma)$ the subspace of $f(z)$ in $M_k(\Gamma)$ which actually vanish at the cusps. Functions in this latter space are called *cusp forms*; for such functions (like the “modular discriminant” $\Delta(z)$), the constant term a_0 in the expansion (2) is zero.

We have already remarked that automorphic forms in general have number-theoretic interest because their Fourier coefficients involve solution numbers of

number-theoretic problems. For example, by relating $\theta_4(z)$ to certain Eisenstein series on $\Gamma(2)$, we obtain Jacobi's remarkable formula

$$r(n, Q_4) = 8 \sum_{\substack{d|n \\ 4|d}} d.$$

Thus the need for analyzing this space $M_k(\Gamma)$ is clearly indicated.

As we shall soon see, the subsequent theory developed by Hecke was so successful that it suggested new ways to look at automorphic forms in number theory as well as immediately providing the tools to solve existing classical problems.

2. *Hecke's theory.* Hecke's key idea was to characterize the properties of an automorphic form in terms of a corresponding Dirichlet series. The most famous Dirichlet series around is, of course, Riemann's zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p < \infty} (1 - p^{-s})^{-1}.$$

So let us first sketch Riemann's original analysis of $\zeta(s)$ which Hecke so brilliantly generalized.

Recall the gamma function identity,

$$\Gamma(s) = \int_0^{\infty} t^s e^{-t} \frac{dt}{t},$$

valid for $\text{Re}(s) > 0$. (In modern parlance, we say that $\Gamma(s)$ is the Mellin transform of e^{-t} at s .) With this identity, we derive the relation

$$\pi^{-s} \Gamma(s) \zeta(2s) = \int_0^{\infty} \left[\frac{\theta(it) - 1}{2} \right] t^s \frac{dt}{t},$$

with θ the classical theta-function already encountered. In other words, $\zeta(2s)$ is essentially the Mellin transform of $\theta(it)$. From this fact, it is a simple matter to derive the desired analytic properties of $\zeta(s)$ in terms of the automorphic properties of $\theta(z)$, and conversely! Here are the key steps:

$$\begin{aligned} \pi^{-s} \Gamma(s) \zeta(2s) &= \int_1^{\infty} t^{s-1} \left(\frac{\theta(it) - 1}{2} \right) dt \\ &\quad - \frac{1}{2} \frac{t^s}{s} \Big|_0^1 + \frac{1}{2} \int_0^1 t^{s-1} \theta(it) dt \\ &= \int_1^{\infty} t^{s-1} \frac{(\theta(it) - 1)}{2} dt - \frac{1}{2s} + \frac{1}{2} \int_1^{\infty} t^{-s-1} \theta\left(\frac{i}{t}\right) dt \\ &\quad \text{(using the change of variable } t \rightarrow 1/t) \\ &= \int_1^{\infty} (t^{s-1} + t^{1/2-(s+1)}) \frac{(\theta(it) - 1)}{2} dt - \frac{1}{2s} - \frac{1}{1-2s} \\ &\quad \left(\text{using the automorphy property } \theta\left(\frac{i}{t}\right) = t^{1/2} \theta(it) \right). \end{aligned}$$

Note that invariance with respect to the substitution $s \rightarrow \frac{1}{2} - s$ is already obvious. To reverse the process and derive the “functional equation”, i.e., automorphy condition of $\theta(z)$ from that of $\zeta(s)$, we require “Mellin inversion”:

$$\frac{\theta(it) - 1}{2} = \frac{1}{2\pi i} \int_{\text{Re}(s)=\sigma>0} t^{-s} \pi^{-s} \Gamma(s) \zeta(2s) ds.$$

By generalizing this proof, Hecke was able to “explain” the symmetry of a large number of Dirichlet series and also paved the way towards finding automorphic forms seemingly everywhere in number theory.

Given a sequence of complex numbers $a_0, a_1, \dots, a_n, \dots$ with $a_n = O(n^c)$ for some $c > 0$, and given $h > 0, k > 0, C = \pm 1$, consider the series

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$\Phi(s) = (2\pi/\lambda)^{-s} \Gamma(s) \phi(s),$$

and the function defined in H by

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/h}.$$

THEOREM 1 (HECKE). *The following two conditions are equivalent:*

(A) $\Phi(s) + a_0/s + C/(k - s)$ is entire, bounded in every vertical strip, and satisfies the functional equation $\Phi(k - s) = C\Phi(s)$;

(B) $f(-1/z) = C(z/i)^k f(z)$.

In other words, the holomorphic function $f(z)$ is automorphic of weight k (for the group of transformations generated by $z \rightarrow z + h$ and $z \rightarrow -1/z$) if and only if its associated Dirichlet series $\sum a_n/n^s$ is “nice”. (We shall often use the term “nice” to describe a Dirichlet series satisfying certain analytic properties similar to $\zeta(s)$.)

The second part of Hecke’s theory answers the question: when does $\phi(s) = \sum a_n/n^s$ have an Euler product expansion of the form $\phi(s) = \prod_{p<\infty} L_p(s)$, with $L_p(s)$ a power series in p^{-s} ? A formal computation shows that $\phi(s)$ factors as

$$\prod_{p \text{ prime}} \sum_{m \geq 0} \frac{a_p m}{p^{ms}}$$

whenever the coefficients a_n are *multiplicative*, i.e., $a_n m = a_n a_m$ if n and m are relatively prime.

Characterizing such multiplicativity is crucial. Indeed, since the coefficients a_n always have number-theoretic significance, it is of great interest to know when knowledge of these a_n ’s can be reduced to knowing a_p for p prime.

Note that when the a_n ’s are *completely* multiplicative, i.e., $a_{nm} = a_n a_m$ for all n and m , the Euler product expansion above reduces to the familiar expression

$$(3) \quad \sum \frac{a_n}{n^s} = \prod_p (1 - a_p p^{-s})^{-1}.$$

Such is, of course, the case for the Riemann zeta-function

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - p^{-s})^{-1}$$

where the Euler product expansion (discovered appropriately by Euler) is tantamount to the fundamental theorem of arithmetic. Another example is provided by the coefficients $a_n = \chi(n)$ with χ a “character of the integers modulo some N ”, i.e., χ is completely multiplicative, of period N , and $\chi(0) = 0$; such a character is called a “Dirichlet character” mod N , and the corresponding series

$$\sum \frac{a_n}{n^s} = \sum \frac{\chi(n)}{n^s} = \prod (1 - \chi(p)p^{-s})^{-1}$$

is a “Dirichlet L -series”.

In both these examples, the Euler factors $L_p(s)$ are of degree 1 in p^{-s} . In general, such an expansion as (2) is too much to ask for; usually we ask for only ordinary multiplicativity, and then the factors $L_p(s)$ turn out to be of the second (not first) degree in p^{-s} .

Hecke’s contribution was to characterize the multiplicativity of the a_n (or $\phi(s)$) intrinsically (and even “locally”) in terms of $f(z)$ by introducing a certain ring of “Hecke operators” $T(p)$ defined in a space of automorphic forms of fixed weight.

THEOREM 2 (HECKE). *Assume, for convenience, that*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

belongs to $S_k(\text{SL}_2(\mathbf{Z}))$ and $a_1 = 1$. Then the a_n ’s are multiplicative (and $\phi(s)$ has an Euler product expansion) if and only if f is an eigenfunction for all the Hecke operators $T(p)$, with $T(p)f = a_p f$. In this case,

$$\phi(s) = \prod L_p(s)$$

with

$$L_p(s) = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

EXAMPLE. Since $S_{12}(\text{SL}_2(\mathbf{Z}))$ is one dimensional, and $T(p)$ preserves this space, the condition $T(p)\Delta = \lambda_p \Delta$ is automatic. Thus one obtains the multiplicativity of the coefficients $\tau(n)$ (conjectured by Ramanujan and first proved by L. J. Mordell).

REMARKS. (1) Hecke’s Theorem 1 really says that an automorphic eigenform of weight k on $\text{SL}_2(\mathbf{Z})$ is indistinguishable from an Euler product of degree 2 with prescribed analytic behavior (and functional equation involving the substitution $s \rightarrow k - s$). This observation sheds a new light on the theory of automorphic forms, since there are many number-theoretical situations where data $\{a_n\}$ leads to a nice L -function, hence by Hecke to an automorphic form. Following up on this idea, A. Weil in 1967 completed Hecke’s theory by

similarly characterizing automorphic forms not just on $\mathrm{SL}_2(\mathbf{Z})$, but also on the so-called congruence subgroups such as

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c \equiv 0 \right\}.$$

(These subgroups have in general many generators, whereas Hecke's theorem deals with automorphic forms only for the groups generated by $z \rightarrow z + \lambda$ and $z \rightarrow -1/z$.)

In this way, Weil was led to an extremely interesting conjecture. By carefully analyzing the zeta-function attached to an elliptic curve E over \mathbf{Q} (with $L_p(s) = (1 - a_p p^{-s} + p^{1-2s})^{-1}$, and $1 + p - a_p$ the number of points on the "reduced curve modulo p "), Weil was able to conjecture that such a zeta-function is the Dirichlet series attached to an automorphic form in some $S_2(\Gamma_0(N))$; cf. [We 1]. In other words, the study of these curves might (perversely) be regarded as a special chapter in the theory of automorphic forms!

(2) Perhaps it is now clear to the reader that an automorphic form $f(z)$ (like an elliptic curve or a quadratic form) should be regarded as a "global object" over \mathbf{Q} , and that the a_p (or the Euler factors $L_p(s)$) comprise local data for f in much the same way that p -adic solutions comprise local data for rational (i.e. "global") solutions of Diophantine equations. This turns out to be the case, but must remain a fuzzy notion until the language of automorphic representations is introduced in Part III.

Note. Two excellent sources on Hecke theory, which we have followed closely, are [Ogg and Rob 3].

C. Artin (and other) L -functions. Around 1840, Dirichlet succeeded in proving the existence of infinitely many primes in an arithmetic progression by replacing (Euler's) analysis of the series $\sum 1/n^s$ by his own analysis of the "Dirichlet L -functions" $L(s, \chi) = \sum \chi(n)/n^s$. Soon afterwards, Riemann focused on such Dirichlet series as functions of a *complex* variable, thereby inspiring a spate of applications of Dirichlet series to number theory in general, and the theory of prime numbers in particular. Finally, in 1870, Dedekind introduced a new kind of zeta-function to study the integers in an arbitrary number field E , i.e., any finite extension of \mathbf{Q} . This kind of zeta-function, now called a Dedekind zeta-function and denoted $\zeta_E(s)$, made it possible to relate the primes of \mathbf{Q} to those of E and to analyze the distribution of primes within E alone.

Despite this widespread use of " L -series" in the nineteenth century, and the concomitant need to generalize these functions further, a full understanding of the arithmetic significance of L -functions awaited twentieth century developments.

1. *Abelian L -functions.* In 1916, Hecke was able to establish the analytic continuation and functional equation of Dirichlet's L -functions and to generalize them to the setting of an arbitrary number field. To describe Hecke's achievement properly, we must first recall how to generalize the family of p -adic fields \mathbf{Q}_p considered in II.A.

Fix a finite extension E of \mathbf{Q} and let O_E denote the ring of integers of E . By a finite "place" or prime v of E we understand a prime ideal \mathfrak{P} in O_E (and we

often confuse the notations v and \mathfrak{P}); by a “fractional ideal” of O_E we understand an O_E -submodule of E with the property that $x\mathfrak{A} \subset O_E$ for some $x \in E^\times$. It is a basic fact that the prime ideals are invertible (in the sense that $\mathfrak{P} \cdot \mathfrak{P}^{-1} = O_E$ for some fractional ideal \mathfrak{P}^{-1}) and that every fractional ideal factors (uniquely) into powers of prime ideals.

Now if x is in E^\times , we define $\text{ord}_{\mathfrak{P}}(x)$ to be the (positive or negative) power of \mathfrak{P} appearing in the factorization of the principal ideal (x) , and we set

$$|x|_{\mathfrak{P}} = |x|_{\mathfrak{P}} = N^{\mathfrak{P} \cdot \text{ord}_{\mathfrak{P}}(x)},$$

with $N^{\mathfrak{P}}$ the cardinality of the field O_E/\mathfrak{P} . By analogy with the case of \mathbf{Q} , we also define a “real” place v of E to be a norm $|x|_v = |\sigma(x)|$, with $\sigma: E \rightarrow \mathbf{R}$ a real embedding. (“Complex” infinite places are defined analogously.) The result is a family of completions E_v of E , one for each prime (or place) v of E . Following the lead of the local-global principle for \mathbf{Q} , we treat all these “finite” or “infinite” places equally.

Recall that a classical Dirichlet character is just a homomorphism of $(\mathbf{Z}/N\mathbf{Z})^\times$ into \mathbf{C}^\times “extended” to \mathbf{Z} by composition with the natural homomorphism $\mathbf{Z} \rightarrow \mathbf{Z}/N\mathbf{Z}$ (and with the convention that $\chi(n) = 0$ if $(n, N) > 1$). The appropriate generalization of such a character to the number field E is called a *Hecke character* (or *groschencharacter*) χ . This is a family of homomorphisms $\chi_v: E_v^\times \rightarrow \mathbf{C}^\times$, one for every place v of E , such that for any x in E^\times (regarded as embedded in each E_v^\times), $\prod_{\text{all } v} \chi_v(x) = 1$. I.e., Hecke characters are “trivial” on E^\times . Implicit here is the fact that all but finitely many of the χ_v are *unramified*, i.e., $\chi_v(x_v) = 1$ for all x_v in E_v^\times such that $|x_v|_v = 1$. (The fact that Dirichlet characters give rise to such Hecke characters is spelled out in [We 3, p. 313]; we shall return to these matters in II.D.1.)

Now we can define Hecke’s abelian L -series attached to $\chi = (\chi_v)$ by

$$L_E(\chi, s) = \sum \frac{\chi(\mathfrak{A})}{N(\mathfrak{A})^s} = \prod_{\mathfrak{P}} (1 - \chi(\mathfrak{P})N^{\mathfrak{P} \cdot s})^{-1}.$$

Here \mathfrak{A} is an (ordinary) ideal of O_E , and $\chi(\mathfrak{A})$ is defined multiplicatively on the ideals relatively prime to those such that χ_v is ramified (the “conductor” of χ). In particular, $\chi(\mathfrak{P}) = 0$ whenever χ_v is ramified; otherwise, $\chi(\mathfrak{P}) = \chi_v(\tilde{\omega}_v)$ with $\tilde{\omega}_v$ in E_v such that $|\tilde{\omega}_v|_v = N^{\mathfrak{P} \cdot -1}$ (a “uniformizing” variable for E_v).

If χ is the unit character, i.e., $\chi_v \equiv 1$ for all v , then $L_E(\chi, s)$ reduces to Dedekind’s zeta-function $\zeta_E(s)$. On the other hand, when $E = \mathbf{Q}$ and χ is of finite order, $L_E(\chi, s)$ reduces to a familiar Dirichlet L -series $L(s, \chi)$. In general, using ingenious and complicated arguments, Hecke was able to derive the analytic continuation and functional equation for all these L -series in 1917.

This settled, a natural arithmetic question was: how does a series like $\zeta_E(s)$ factor into L -series involving *only* the field \mathbf{Q} ? Partly in an attempt to solve this problem, E. Artin was led (around 1925) to define yet another new L -series.

2. *Nonabelian L-functions.* Suppose K is a number field, and E is a finite Galois extension of K with Galois group $G = \text{Gal}(E/K)$. By a *representation* of G we understand a homomorphism σ of G into $\text{GL}(V)$, the group of invertible linear transformations of a complex vector space of dimension n .

Given a finite place \mathfrak{P} of K , we say a prime \mathfrak{B} of O_E lies over \mathfrak{P} (or divides \mathfrak{P}) if \mathfrak{B} appears in the factorization of $\mathfrak{P}O_E$ into prime ideals of O_E . Given such a pair $\mathfrak{B}/\mathfrak{P}$, the following “local” objects are defined, each depending on \mathfrak{B} only up to conjugation in G :

(1) The decomposition group $D_{\mathfrak{B}} = \{g \in G: g(\mathfrak{B}) = \mathfrak{B}\}$;

(2) The inertia subgroup

$$I_{\mathfrak{B}} = \{g \in D_{\mathfrak{B}}: g(x) = x \pmod{\mathfrak{B}} \text{ for all } x \in O_E\};$$

(3) The Frobenius automorphism $\text{Fr}_{\mathfrak{B}}$ generating the cyclic group $D_{\mathfrak{B}}/I_{\mathfrak{B}} \approx \text{Gal}(O_E/\mathfrak{B}: O_K/\mathfrak{P})$. When $I_{\mathfrak{B}} = \{1\}$, we call \mathfrak{P} unramified in E . In this case, $\text{Fr}_{\mathfrak{B}}$ represents a conjugacy class in G . Because almost all \mathfrak{P} are unramified in this sense, and because knowledge of $\text{Fr}_{\mathfrak{B}}$ completely determines the factorization type of such \mathfrak{P} (cf. our Introduction), it is natural to focus attention on these $\text{Fr}_{\mathfrak{B}}$. In general, if E is abelian over K , i.e., $G = \text{Gal}(E/K)$ is abelian, then $D_{\mathfrak{B}}$, $I_{\mathfrak{B}}$ and $\text{Fr}_{\mathfrak{B}}$ depend only on \mathfrak{P} (and are denoted $D_{\mathfrak{P}}$, $I_{\mathfrak{P}}$ and $\text{Fr}_{\mathfrak{P}}$ accordingly). In particular, $\text{Fr}_{\mathfrak{P}}$ reduces to a distinguished element (as opposed to a conjugacy class) of G .

Returning to our representation $\sigma: G \rightarrow \text{GL}(V)$, let $V_{\mathfrak{B}}$ denote the subspace of V formed by vectors invariant by $\sigma(I_{\mathfrak{B}})$. Then $\sigma(\text{Fr}_{\mathfrak{B}})$ is defined unambiguously on $V_{\mathfrak{B}}$, and the “Euler factor”

$$L_{\mathfrak{P}}(\sigma, s) = \left[\det(I - \sigma(\text{Fr}_{\mathfrak{B}})N^{\mathfrak{P}-1} |_{V_{\mathfrak{B}}}) \right]^{-1}$$

depends only on \mathfrak{P} , not \mathfrak{B} . Note that for almost all \mathfrak{P} , $I_{\mathfrak{B}} = \{1\}$, $V_{\mathfrak{B}} = V$, and hence $L_{\mathfrak{P}}(\sigma, s)$ is of degree n in $(N^{\mathfrak{P}})^{-s}$.

Now we can finally define Artin’s L -function attached to σ by the product

$$L(\sigma, s) = L_{E/K}(\sigma, s) = \prod_{\mathfrak{P}} L_{\mathfrak{P}}(\sigma, s),$$

convergent for $\text{Re}(s) > 1$.

THEOREM 1 (ARTIN). *Suppose E is abelian over K , and $\sigma: G \rightarrow \mathbb{C}^{\times}$ is a character. Then there exists a Hecke character $\chi = \{\chi_v\}_v$ of K (not E) such that $L_{E/K}(\sigma, s) = L(\chi, s)$ (Hecke’s abelian L -series for K).*

In order to prove this theorem, Artin formulated and proved his celebrated “reciprocity law” for abelian extensions of number fields. For a leisurely account of these matters, see [Tate]. Because Artin and Hecke L -series coincide in the case of abelian extensions, the terminology “abelian L -functions” for Hecke’s $L(s, \chi)$ seems apt.

THEOREM 2 (ARTIN-BRAUER). *In general, $L(\sigma, s)$ extends to a meromorphic function of \mathbb{C} with functional equation*

$$L(\sigma, s) = \varepsilon(\sigma, s)L(\sigma^{\vee}, 1 - s),$$

σ^{\vee} the contragredient representation $\sigma(g^{-1})'$.

The proof of this theorem relies on Theorem 1 together with R. Brauer’s (1947) factorization

$$L_{E/K}(\sigma, s) = \prod L_{E/K_i}(\sigma_i, s)^{n_i};$$

here each K_i is intermediate between K and E , σ_i is a representation of $\text{Gal}(E/K_i)$ of dimension 1, and n_i is an integer.

THEOREM 3 (ARTIN). *Let H denote the (finite) set of all irreducible representations of G (listed up to isomorphism). Then*

$$\zeta_E(s) = \prod_{\sigma \in H} L_{E/K}(\sigma, s)^{\dim(\sigma)}.$$

This is the sought-after factorization of $\zeta_E(s)$ alluded to at the end of paragraph II.C.1. Note that when E is *abelian* over K , and $K = \mathbf{Q}$, Theorems 3 and 1 together imply that $\zeta_E(s)$ factors as the product of Hecke L -series, a nontrivial assertion involving the theory of cyclotomic fields (in particular, the Kronecker-Weber theorem asserting that every abelian extension of \mathbf{Q} embeds in some cyclotomic field).

Question. When $\dim(\sigma) > 1$, what is the nature of these *nonabelian* L -series $L(\sigma, s)$? In particular, if $\dim(\sigma) = 2$, does $L(\sigma, s)$ have any relation to Hecke's "automorphic" Dirichlet series $\phi_f(s) = \sum a_n/n^s$ (if not with the abelian L -series $L(\chi, s)$)?

CONJECTURE (ARTIN). Suppose $\dim(\sigma) > 1$, and σ is irreducible, i.e., V has no proper invariant subspaces under $\sigma(G)$. Then $L_{E/K}(\sigma, s)$ is actually entire.

Note that the truth of Artin's Conjecture is at least consistent with the known analytic behavior of the automorphic L -functions $L(s, f)$ (cf. Hecke's Theorem 2 in II.B.2). We shall return to these matters in earnest in Parts III and IV.

CONCLUDING REMARK. The "constant" $\varepsilon(\sigma, s)$ appearing in the functional equation for $L(\sigma, s)$ is defined by piecing together some global notions (the Artin "conductor of σ ") with a product of gamma functions. But following the lead of our local-global principle, one should attempt to define $\varepsilon(\sigma, s)$ instead as the product of purely local factors $\varepsilon(\sigma_v, s)$. Eventually this was accomplished by Langlands (with some finishing touches by Deligne). As we shall see, this accomplishment made possible a serious attack on the above questions (at least for $\dim(\sigma) = 2$) and helped to develop the program this article eventually describes.

Note. Our exposition here closely follows the first few sections of Cartier's Bourbaki talk [Cart 1].

D. Group representations in number theory. We come at last to the fourth and final ingredient for the soup we shall mix up in Part III.

If G is a group, a (unitary) representation of G is a homomorphism π from G to the group of invertible (unitary) operators on some (Hilbert) space V , not necessarily finite dimensional. If G is a topological group, continuity assumptions are also imposed on π , but we shall ignore them here. Examples of representations encountered thus far in this survey include:

(1) The case when V is the one-dimensional space \mathbf{C} ; in this case, a representation on V is simply a character—a homomorphism from G to (the torus in) \mathbf{C}^\times . A Hecke character, for example, is a collection (χ_v) of 1-dimensional representations of E_v^\times .

(2) The case when V is finite dimensional and G is a finite group; the Artin L -functions $L(\sigma, s)$, for example, are attached to n -dimensional representations σ of $G = \text{Gal}(E/K)$.

The use of group representations in systematizing and resolving diverse mathematical problems is certainly not new, and the subject has been ably surveyed in several recent articles, notably [Gross and Mackey]. The reader is strongly urged to consult these articles, especially for their reformulation of harmonic analysis as a chapter in the theory of group representations.

In harmonic analysis, as well as in the theory of automorphic forms, the fundamental example of a (unitary) representation is the so-called "right regular" representation of G , defined in a space of functions $\{\phi\}$ on G by the formula $(R(g)\phi)(x) = \phi(xg)$.

Our interest here is in the role representation theory has played in the theory of automorphic forms. We focus on two separate developments, both of which are eventually synthesized in the Langlands program, and both of which derive from the original contributions of Hecke already described.

1. *Tate's adelic treatment of Hecke's L -series.* In his 1950 Princeton thesis, J. Tate showed how to use representations (or rather characters) of adèle groups to reformulate and reprove Hecke's (complicated) results on the functional equation of his "abelian" L -series.

What is an adèle group? It is really just a device for simultaneously considering all the completions of an object such as the rational field \mathbf{Q} . In this context, the adèle ring \mathbf{A} is the subgroup of the infinite product $\prod_{p < \infty} \mathbf{Q}_p$ consisting of those sequences $\{x_p\}$ with $|x_p|_p \leq 1$ for all but a finite number of primes p . The *idele* group \mathbf{A}^\times is defined similarly as the subgroup of the (multiplicative) group $\prod_{p < \infty} \mathbf{Q}_p^\times$ consisting of sequences $\{y_p\}$ with $|y_p|_p = 1$ for all but finitely many places p . In other words, the adèles and ideles are *restricted* direct products of p -adic objects. For a general number field in place of \mathbf{Q} , similar definitions apply with v in place of p .

Note that the assignment of an idele or adèle group to a number field F is entirely consistent with the local-global principle of II.A in that all local objects are introduced on an equal footing and the goal is to analyze the global object F . Pushing this principle to the extreme, Tate used this notion to define and analyze Hecke's "abelian L -functions" purely in local terms, as we shall now explain.

Let \mathbf{A}_F (resp. \mathbf{A}_F^\times) denote the adèles (resp. ideles) of F . The topology on \mathbf{A} (resp. \mathbf{A}^\times) is defined in such a way that every continuous character ψ of \mathbf{A} is of the form $\psi(x) = \prod_v \psi_v(x_v)$ with ψ_v a continuous character of F_v for each place v , and for all but finitely many v , $\psi_v(x_v) = 1$ whenever $|x_v|_v \leq 1$. Similarly, any character χ of \mathbf{A}^\times is of the form $\prod_v \chi_v(x_v)$, with χ_v *unramified* for almost all v , i.e., $\chi_v(x_v) = 1$ whenever $|x_v|_v = 1$. Thus a character χ which is trivial on the so-called principle ideles F^\times (embedded diagonally in \mathbf{A}^\times) is the same thing as a Hecke character (χ_v) of E . In this case, we also call χ an *idele class* character, since it descends to a character of the so-called idele class group $\mathbf{A}^\times/F^\times$.

Now fix a character $\psi = \prod \psi_v$ of \mathbf{A} which is trivial on F (again embedded diagonally in \mathbf{A}). Given a Hecke character $\chi = \prod \chi_v$, Tate defined, for each

nice “Schwartz function” f_v on the field F_v , a “local zeta-function”

$$\zeta(f_v, \chi_v, s) = \int_{F_v^*} f_v(x) \chi_v(x) |x|^{s-1} dx.$$

Using a local Fourier transform \hat{f}_v (defined in terms of the local character ψ_v), Tate then proved that each $\zeta(f_v, \chi_v, s)$ has an analytic continuation and functional equation of the form

$$\zeta(f_v, \chi_v, s) = \gamma(\chi_v, \psi_v, s) \zeta(\hat{f}_v, \chi_v^{-1}, 1 - s),$$

with $\gamma(\chi_v, \psi_v, s)$ a meromorphic function of s which is independent of f_v . In fact, let

$$L(\chi_v, s) = (1 - \chi_v(\tilde{\omega}_v) N^{\mathfrak{q}^{-s}})^{-1}$$

if χ_v is unramified, and 1 otherwise. Then $\zeta(f_v, \chi_v, s)/L(\chi_v, s)$ is entire for all f_v , and equals 1 for appropriately chosen f_v ; moreover,

$$\gamma(\psi_v, s) = \frac{\varepsilon(\chi, \psi_v, s)^{-1} L(\chi_v^{-1}, 1 - s)}{L(\chi_v, s)}$$

with $\varepsilon(\chi_v, \psi_v, s) = 1$ whenever χ_v (also ψ_v) is unramified.

Now return to the global setting and consider the (global) Hecke L -series

$$L(\chi, s) = \prod_v L(\chi_v, s).$$

By considering global zeta-functions of the form

$$\zeta(f, \chi, s) = \int_{\mathbb{A}^x} f(x) \chi(x) |x|^{s-1} dx = \prod_v \zeta(f_v, \chi_v, s),$$

Tate was *easily* able to prove that $L(\chi, s)$ has an analytic continuation and functional equation of the form

$$L(s, \chi) = \varepsilon(s, \chi) L(1 - s, \chi^{-1}) \quad \text{with } \varepsilon(s, \chi) = \prod_v \varepsilon(\chi_v, \psi_v, s).$$

In addition to clarifying and simplifying Hecke’s work, Tate was thus also able to give a purely local interpretation to the “constant” ε appearing in Hecke’s functional equation.

These ideas and others are vastly generalized by Langlands in the “Langlands program”. The immediate impetus and inspiration for this program actually came from Langlands’ general theory of Eisenstein series (cf. [Art]). The sequence of events seems to have unfolded like this: a careful analysis of the “constant term” of these general Eisenstein series first suggested the definition of the general automorphic L -functions $L(s, \pi)$ (discussed in Part III), then the general conjectures (discussed in Part IV), and finally the purely local definition of the constants in the functional equations of Artin’s *non-abelian* L -series (analogous to Tate’s treatment of the abelian case). In fact, the existence and properties of the local ε -factors “on the Galois side” were predicted by the newly found properties of the ε -factors “on the representation theory side”; cf. Part III. It is interesting to note that here, as in Tate’s work

and its generalizations, most of the work in proving the global results goes into the local theory; for earlier important work on the “Galois” ε -factors, see [Dwork].

Finally, as we shall see in Part III, Hecke characters are themselves examples of automorphic representations. Thus most of these results of Tate, Hecke and Artin are subsumed simultaneously in Langlands’ theory.

Notes. There are now several good expositions of Tate’s original work, [Rob 2] being one of them. Tate’s thesis itself makes for surprisingly pleasant reading; cf. [Cas Fro].

2. *Automorphic forms as group representations.* How do classical automorphic forms amount to special examples of infinite-dimensional representations—first of the group $SL_2(\mathbf{R})$, then of the so-called adèle group of GL_2 ?

For convenience, we shall consider only automorphic forms of even integral weight for the full modular group $\Gamma = SL_2(\mathbf{Z})$; for details and more examples, the reader is referred to [Ge 1].

The connection between automorphic forms and infinite-dimensional representations seems first to have been made explicit in [Ge Fo]. It starts with the observation that the stabilizer in $SL_2(\mathbf{R})$ of the point i in the upper halfplane H is the rotation group $SO_2(\mathbf{R})$, whence the identification

$$H \approx SL_2(\mathbf{R})/SO_2(\mathbf{R}).$$

If $f(z)$ is an automorphic form of weight k for $SL_2(\mathbf{Z})$ it defines a function ϕ_f on $SL_2(\mathbf{R})$ through the formula

$$(1) \quad \phi_f(g) = (ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right) \quad \text{if } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This function can be shown to satisfy the following properties:

$$(0) \quad \phi(g\gamma(\theta)) = e^{-ik\theta}\phi(g) \quad \text{for } r(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix};$$

(i) $\phi(\gamma g) = \phi(g)$ for all ϕ in $SL_2(\mathbf{Z})$;

(ii) there is a second-order differential operator Δ —called the Laplace or Casimir operator on $SL_2(\mathbf{R})$ —such that

$$\Delta\phi = -\frac{1}{4}k(k - 2)\phi.$$

Indeed the automorphy property of $f(z)$ translates into condition (i), and the holomorphy of f translates into the eigenfunction condition (ii). The technical regularity condition on f “at the cusps” corresponds to a boundedness condition on ϕ , and if f actually “vanishes” at the cusps, then ϕ satisfies a “cuspidal” condition which puts it in a distinguished subspace L_0^2 of $L^2(SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R}))$. Here L^2 denotes square-integrable functions with respect to the natural Haar measure on $SL_2(\mathbf{R})$ pushed down to the quotient space $SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R})$; the “cuspidal” condition is included in the following hypothesis:

(iii) $\phi \in L^2$ and the “zeroth Fourier coefficient”

$$\int_{\mathbf{Z}\backslash\mathbf{R}} \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}g\right) dx$$

vanishes.

Conversely, if $\phi(g)$ is any function on $SL_2(\mathbf{R})$ satisfying these four conditions, then ϕ is of the form ϕ_f for some f in $S_k(\Gamma)$. In other words, $S_k(\Gamma)$ is isomorphic to a certain subspace $A_k(\Gamma)$ of the space of “cusp” forms $L_0^2(SL_2(\mathbf{Z}) \backslash SL_2(\mathbf{R}))$.

To obtain a more general notion of “automorphic cusp form on $SL_2(\mathbf{R})$ ” it is natural to weaken (ii) to the condition that ϕ be *any* eigenform of Δ . In this way we recover, for example, the “new” real-analytic “wave forms” which H. Maass discovered in 1949 in connection with his analysis of the functional equation of the Dedekind zeta-function of real (as opposed to imaginary) quadratic extensions of \mathbf{Q} ; cf. §2 of [Ge 1] for more details.

Having brought into play the group $SL_2(\mathbf{R})$, the next step is to bring into play its representation. (It is also natural to weaken condition (0) and simply require \emptyset to be “right $SO(2)$ -finite”.)

The *right regular representation* R of $SL_2(\mathbf{R})$ on the space L_0^2 is defined by

$$R(g)\phi(h) = \phi(hg), \quad g, h \text{ in } SL_2(\mathbf{R}).$$

We recall this is a unitary representation of G because the mapping $g \mapsto R(g)$ is a homomorphism from G to the group of unitary operators on L_0^2 . Moreover, this representation is related to the differential operator Δ since it commutes with Δ , i.e.,

$$\Delta R\phi = R\Delta\phi, \quad \text{for all nice } \phi.$$

By a general principle of representation theory, such *invariant* operators Δ may be used to decompose the representation R into invariant subspaces. More precisely, the closure of each eigenspace \mathcal{L} of Δ is then *invariant* for the action of R , i.e., $R(g)(\mathcal{L}) \subset \mathcal{L}$ for all g in $SL_2(\mathbf{R})$. In particular, R is the direct sum of representations of $SL_2(\mathbf{R})$ in the various eigenspaces \mathcal{L} .

Since all the irreducible unitary representations of $SL_2(\mathbf{R})$ are known, it is natural to ask which of them occurs in these eigenspaces, and with what multiplicity.

EXAMPLE. Fix f in $S_k(SL_2(\mathbf{Z}))$, and let \mathcal{L}_f denote the subspace of $L_0^2(SL_2(\mathbf{Z}) \backslash SL_2(\mathbf{R}))$ generated by ϕ_f and its right translates by elements of $SL_2(\mathbf{R})$. Suppose, in addition, that f is an eigenfunction for all the Hecke operators $T(p)$. Then \mathcal{L}_f is an irreducible subspace of L_0^2 realizing the so-called discrete series representation of $SL_2(\mathbf{R})$ of weight k .

This example shows that automorphic cusp forms are one and the same thing as irreducible unitary representations of $SL_2(\mathbf{R})$ which are “automorphic cuspidal”, i.e., equivalent to some subrepresentation of R in L_0^2 . In other words, the construction of automorphic forms amounts to knowing how the right regular representation R decomposes. Thus it is natural to generalize this notion further to an arbitrary Lie group G and discrete subgroups Γ such that $\Gamma \backslash G$ has finite volume. In practice, one deals primarily with “semisimple” groups G and “arithmetic” subgroups Γ . For example, if G is the real symplectic group $Sp_n(\mathbf{R})$, and $\Gamma = Sp_n(\mathbf{Z})$, a generalization of the so-called “Siegel modular forms” (studied by C. L. Siegel) is obtained; cf. [Langlands 1]. In general, much progress has been made in analyzing qualitative features of the decomposition of the regular representation R of G in the space L_0^2 . In particular, one knows there are only finitely many invariant subspaces of L_0^2

which are equivalent (as G -modules) to a given “automorphic cuspidal representation” π (similarly any space of generalized cusp forms, of “fixed weight” for Γ , is finite dimensional).

Despite the appeal of the program just described, recalling the local-global principle should give us pause for thought. By working with a group like $SL_2(\mathbf{R})$, we have put special emphasis on \mathbf{R} as a completion of \mathbf{Q} , whereas we really should be paying equal attention to all the p -adic fields \mathbf{Q}_p . Indeed by dealing only with $SL_2(\mathbf{R})$, we cannot hope to naturally view $f(z)$ as a “global object”, or to throw light on the Euler product expansion which can exist for the Dirichlet series of $f(z)$.

What is clearly indicated now is that we view groups like $SL_2(\mathbf{R})$ as the infinite component of some *global* object. Following the lead of imbedding \mathbf{R} as the infinite component of $\mathbf{A}_{\mathbf{Q}}$, we set, for each prime p ,

$$G_p = GL_2(\mathbf{Q}_p), \quad K_p = GL_2(O_p),$$

where $O_p = \{x \in \mathbf{Q}_p : |x|_p \leq 1\}$, and we write G_∞ for $GL_2(\mathbf{R})$. Then the corresponding adelic group $G_{\mathbf{A}}$ is defined to be the restricted direct product

$$G_{\mathbf{A}} = GL_2(\mathbf{A}) = \prod'_{p < \infty} G_p,$$

where “restricted” here means we are including only the sequences (g_p) for which g_p belongs to K_p for all but finitely many p . The group $G_{\mathbf{Q}} = GL_2(\mathbf{Q})$ is embedded as a discrete subgroup of $G_{\mathbf{A}}$ via the diagonal embedding.

Now suppose, as before, that $f(z)$ belongs to $S_k(SL_2(\mathbf{Z}))$. By modifying formula (1), we can “lift” $f(z)$ all the way up to a function $\phi_f(g)$ on $G_{\mathbf{A}}$ such that

$$\phi(\gamma g) = \phi(g), \quad \text{for all } \gamma \text{ in } G_{\mathbf{Q}},$$

and $\phi(zg) = \phi(g)$, for all z in the center $Z_{\mathbf{A}} = \{ \begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} : a \in \mathbf{A}^\times \}$ of $G_{\mathbf{A}}$. Moreover, $\phi_f(g)$ satisfies a cuspidal condition like (iii); we refer the reader to [Ge 1] for a careful discussion of the approximation theorem which makes this lifting possible and well defined. The switch from SL_2 to GL_2 was made for reasons of convenience which need not concern the nonexpert.

Where do we stand now? We again have a right regular representation R of $GL_2(\mathbf{A})$, this time on $L^2_0(Z_{\mathbf{A}}G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$, and the functions ϕ_f just described lie in this space. So we may again ask, for a given f , which irreducible unitary representations of $G_{\mathbf{A}}$ occur in the representation R restricted to the subspace generated by each ϕ and its right translates. It is known that:

(i) Every irreducible unitary representation π of $G_{\mathbf{A}}$ is of the form $\otimes_p \pi_p$, with π_p an irreducible unitary representation of G_p for each p , and for almost all p , π_p has a vector fixed for the action of $\pi_p(K_p)$, i.e., π_p is *unramified*. (This is in complete analogy to the description of an idele class character χ as a Hecke character $\prod \chi_p$.)

(ii) The classical Hecke operators $T(p)$ can be described purely *locally* as convolution operators on the G_p -component of the function ϕ_f on $G_{\mathbf{A}}$.

(iii) If $T(p)f = a_p f$ for all p , then ϕ_f generates an *irreducible* subspace \mathcal{L}_f of $L^2_0(Z_{\mathbf{A}}G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$, and the representation $\pi = \otimes \pi_p$ realized in this subspace can be described purely locally in terms of the eigenvalues a_p (and the weight k).

The point is that we have finally succeeded in viewing automorphic forms as global objects which can be analyzed by “local objects” (Hecke operators, for example). Moreover, we have indicated that these notions might make sense for groups more general than GL_2 , and that all “places” should be treated on an equal footing. Thus we are ready—at last—to introduce the general notion of an “automorphic representation” and to describe Langlands’ theory.

We note that classical automorphic forms on all the so-called congruence subgroups of $SL_2(\mathbf{Z})$ can also be lifted to functions on $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$. So by working “adelically” with GL_2 , we are actually simultaneously treating the theory of automorphic forms for *all* these discrete subgroups. For a thorough discussion of these and related matters, see [Ge 1, Rob 1 and G GPS].

III. AUTOMORPHIC REPRESENTATIONS

In this part, G will denote the group GL_n regarded as an “algebraic group over the number field F ”. For all practical purposes, this simply means we can “complete” G at any place v of F to form the groups $G_v = GL_n(F_v)$ and then “adelize” G by forming the restricted direct product $G_{\mathbf{A}} = \prod_v G_v$, just as we did for GL_2 and $GL_1 (= F^{\times})$. In the case of GL_n , the product $\prod_v G_v$ is restricted with respect to the subgroups $K_v = GL_n(O_v)$, with $O_v = \{x \in F_v : |x|_v \leq 1\}$. By G_F we denote the group $G_F = GL_n(F)$ regarded as a discrete subgroup of $G_{\mathbf{A}}$ via the diagonal embedding $\gamma \rightarrow (\gamma, \dots, \gamma, \dots)$.

Although GL_n is almost sufficient for the purposes of this expository article, it should be stressed that we could just as well be dealing with an arbitrary “connected reductive algebraic group G defined over F ” (as we shall in fact do at the end of Part IV).

A. Some definitions. Let L_0^2 denote the Hilbert space of square-integrable functions ϕ on $G_F \backslash G_{\mathbf{A}}$ satisfying a certain cuspidal condition. When $n = 1$, this condition is vacuous; when $n = 2$, it generalizes the condition $a_0 = 0$ for functions ϕ corresponding to classical automorphic forms $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ (cf. condition (iii) in II.D.2). By R_0 we denote the right regular representation of $G_{\mathbf{A}}$ in L_0^2 given by $R_0(g)\phi(h) = \phi(hg)$.

Suppose π is an irreducible unitary representation of $G_{\mathbf{A}}$ in some space H_{π} . Then π can be factored as a restricted direct product $\pi = \otimes \pi_v$, with each π_v an irreducible unitary representation of $G_v = GL_n(F_v)$. This is analogous to the factorization $\prod X_v$ encountered for Hecke characters, and to the factorization $\pi_f = \otimes \pi_p$ discussed in Part II.D.2. In particular, for almost all v , π_v is *unramified*; this means that its space H_v contains (a one-dimensional space of) vectors which are left fixed by the group of operators $\pi(k_v)$, k_v in K_v . The product $\otimes_v \pi_v$ is restricted in the sense that its space consists of vectors $w = \otimes w_v$ with the property that w_v is K_v -fixed for almost all v . Thus the representation $\pi(g)$ is well defined by the formula

$$\pi(g)w = \prod_v \pi_v(g_v)w_v$$

since $\pi_v(g_v)w_v$ belongs to K_v for almost every v .

If there exists an isomorphism A between H_{π} (the space of π) and some subspace V_{π} of L_0^2 such that $A\pi(g) = R(g)A$ for all g in $G_{\mathbf{A}}$, then we say π is equivalent to some subrepresentation of R_0 , or more simply, that π “appears” in R_0 .

DEFINITION. An irreducible unitary representation of $G_{\mathbf{A}}$ is called a *cuspidal automorphic representation of G* if it appears in the right regular representation R_0 of G .

REMARKS. (i) In order to avoid further technical discussion, we have overlooked the fact that $G_F \backslash G_{\mathbf{A}}$ does not quite have finite volume, and thus that L_0^2 does not quite possess minimally invariant subspaces V_{π} . These technical problems are overcome by considering invariance with respect to the central subgroup

$$Z = \left\{ \begin{bmatrix} z & & 0 \\ & \ddots & \\ 0 & & z \end{bmatrix} : z \in \mathbf{A} \right\};$$

the reader can consult [Ge 1 or Art] for details.

(ii) For similar reasons we have made our notion of automorphic representation more restrictive than it should be. Indeed, we should not be restricting ourselves to cuspidal functions, and we also should not be insisting that π necessarily occur “discretely” in any right regular representation. The correct notion of automorphic representation, together with a discussion of the technical problems which ensue, is discussed in [Langlands 5 and Bo Ja].

EXAMPLES OF AUTOMORPHIC REPRESENTATIONS. (1) An automorphic representation of GL_1 is just a character $\chi = \prod \chi_v$ on the idele class group $F \backslash \mathbf{A}$, and $L(s, \chi) = \prod (1 - \chi_v(\tilde{\omega}_v) Nv^{-s})^{-1}$ is the L -function attached to this automorphic representation. Here, and henceforth, Nv denotes the cardinality of the finite field O_v/P_v , the ring of integers in F_v modulo the prime ideal $P_v = \{x \in O_v : |x| \leq 1\}$.

(2) In II.D.2, we saw that a classical automorphic form in $S_k(SL_2(\mathbf{Z}))$, with

$$T(p)f = a_p f \quad \text{for all } p,$$

uniquely determines an automorphic cuspidal representation $\pi_f = \otimes \pi_p$ of GL_2 over \mathbf{Q} . Conversely, every automorphic cuspidal representation $\pi = \otimes \pi_p$ determines some classical automorphic form $f(z)$, though not necessarily for the full modular group $SL_2(\mathbf{Z})$, and not necessarily holomorphic (as opposed to real analytic). Moreover, for f as above, the p th local factor in the Euler product expansion of $\phi_f(s) = \sum a_n n^{-s}$ is

$$L_p(s) = (1 - a_p p^{-s} + p^{k-1-2s})^{-1} = [(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})]^{-1}.$$

In both these examples, important information about global objects (χ or $f(z)$) is stored by local objects (χ_v or a_p). Indeed, it is precisely this fact which makes it possible to introduce an L -function in local terms and in a uniform manner independent of the underlying group.

Let us now explain how this works in general.

B. Local invariants. Given an arbitrary irreducible unitary representation $\pi = \otimes \pi_v$, how can we attach to π_v local data which will simultaneously characterize π_v and feed neatly into a local Euler factor $L(s, \pi_v)$ —even when π is not necessarily automorphic? The answer to this question involves the so-called theory of spherical functions and unramified representations of G_v .

Fix a finite place v of F , and suppose π_v is an unramified representation of G_v . Then the theory of spherical functions assigns to π_v an element A_v of the complex torus T^n which is unique up to the natural action of the permutation group S_n on T^n .

Roughly speaking, the procedure is this: let H_v^0 denote the space of $GL_n(O_v)$ -fixed vectors w_v in the space of π_v , and let \mathcal{H}_v denote the convolution algebra of locally constant, $GL_n(O_v)$ -bi-invariant, and compactly supported complex-valued functions on G_v (called the Hecke algebra); then the formula

$$\pi_v(f)w_v = \int_{G_v} f(g)\pi(g)w_v dg = \chi_{\pi_v}(f)w_v$$

defines a representation of \mathcal{H}_v on the one-dimensional space H_v^0 , and the resulting character $f \rightarrow \chi_{\pi_v}(f)$ defines (via the so-called Satake isomorphism) an element A_v of T^n (unique up to permutation); details are discussed in §IV of [Cart 2] and §§6, 7 of [Bo]. In fact, π_v is itself determined by (the class of) A_v . Thus there results an injection $\pi_v \mapsto A_v$ taking unramified representations of G_v to “semisimple” (i.e., diagonalizable) conjugacy classes in the group $GL_n(\mathbb{C})$.

Let us return now to the global situation. Given $\pi = \otimes \pi_v$, we let S_π denote the finite set of places v outside of which π_v is unramified, and we consider the family of conjugacy classes $\{A_v\}$, $v \notin S_\pi$. The point is that this construction generalizes the assignment of conjugacy classes which is implicit in the classical theory of Hecke, and yet it makes sense for all (not necessarily automorphic) representations π of $GL_n(\mathbb{A})$.

For example, if $n = 1$ and π_v is an unramified character χ_v of F_v^\times , then A_v is simply $\chi_v(\tilde{\omega}_v)$, the value of χ_v at any local uniformizing variable. Also, if $\chi = \prod \chi_v$ is actually automorphic, i.e., trivial on F^\times , then the family $\{A_v\}$, $v \notin S_\pi$, actually determines χ uniquely.

Now suppose $n = 2$, and $\pi_f = \otimes \pi_p$ is an automorphic cuspidal representation of $GL_2(\mathbb{A})$ corresponding to the classical cusp form $f(z) = \sum a_n e^{2\pi i n z}$ in $S_k(SL_2(\mathbb{Z}))$. If $T(p)f = a_p f$ for all p , then

$$A_p = \begin{bmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{bmatrix} \text{ in } GL_2(\mathbb{C}),$$

with $\alpha_p \beta_p = p^{k-1}$ and $\alpha_p + \beta_p = a_p$. In particular, the eigenvalues a_p completely determine the local representations π_p , and it turns out that the family of classes $\{A_p\}$ completely determines f (or π).

In general, for GL_n , it still turns out that the local data $\{A_v\}$ determine the global representation π uniquely *provided* π is automorphic. (This is a nontrivial recent result of Jacquet, Piatetski-Shapiro and Shalika [JPSS 2].) Thus we are witnessing a powerful variation of the local-global principle at work in the context of general automorphic representations. Simultaneously, we are obtaining a handle on introducing general L -functions and comparing automorphic representations of different groups, both cornerstones of the Langlands program we shall now finally describe.

IV. THE LANGLANDS PROGRAM

How does Langlands' theory not only tie together the strands woven in Part II but also develop the general notions of Part III (automorphic representations and L -functions) beyond their classical origins?

A. Preliminary L -functions. Langlands' theory begins by attaching an L -function to an arbitrary irreducible unitary representation $\pi = \otimes \pi_v$ of G_A . (Here, as in Part III, G is the group GL_n , but the more adventurous reader can imagine it to be an arbitrary reductive algebraic group.)

Given π , let S_π again denote the finite set of places v of F outside of which π_v is unramified, and for each $v \notin S_\pi$ let A_v denote the semisimple conjugacy class

$$A_v = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}$$

in $GL(n, \mathbb{C})$ corresponding to π_v as in III.B. Then consider the Euler product

$$L_{S_\pi}(s, \pi) = \prod_{v \notin S_\pi} L(s, \pi_v)$$

with

$$L(s, \pi_v) = [\det(I - [A_v]Nv^{-s})]^{-1}.$$

This is an Euler product of degree n in the sense that each Euler factor $L(s, \pi_v)$ is of the form $P^{-1}(Nv^{-s})$ with P a polynomial of degree n and $P(0) = 1$. The infinite product can be shown to converge for $\text{Re}(s)$ sufficiently large.

THEOREM. *Suppose $\pi = \otimes \pi_v$ is an arbitrary irreducible representation of G_A . Then for all v one can define Euler factors $L(s, \pi_v)$ of degree $\leq n$, and local factors $\epsilon(s, \pi_v)$, such that $\epsilon(s, \pi_v)$ is 1 for almost all v , and*

$$L(s, \pi_v) = \det(I - [A_v]Nv^{-s})^{-1}$$

whenever v is unramified. Moreover, if π is actually automorphic cuspidal, then the Euler product

$$L(s, \pi) = \prod_v L(s, \pi_v),$$

initially defined only in some right half-plane, satisfies the following properties:

- (i) *it extends to an entire function of \mathbb{C} (unless $n = 1$ and π is the trivial character, in which case $L(s, \pi)$ has a pole);*
- (ii) *it satisfies the functional equation*

$$L(s, \pi) = \epsilon(s, \pi)L(1 - s, \tilde{\pi}),$$

with $\tilde{\pi}$ the representation "contragredient" to π , and

$$\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v)$$

REMARKS. (i) This theorem was first proved by Jacquet and Langlands in [JL] for the case $n = 2$. The general case appears in [GoJa]. We note that when $n = 1$, this theorem reduces to the work of Tate described in Part II.D.1. In case $n = 2$ it represents a vast reformulation and generalization of Hecke’s work; in particular, it finally sheds light on the Euler product expansion of Hecke’s Dirichlet series

$$\phi_f(s) = \sum \frac{a_n}{n^s} = (2\pi)^s \Gamma(s)^{-1} L(s', \pi_f)$$

and interprets the constant in the functional equation of $\phi_f(s)$ in terms of the local groups G_v . This last point underscores one of the key contributions of the original work of Jacquet and Langlands and will be discussed presently.

(ii) It is natural and important to ask if this theorem has a converse. In other words, suppose that $\{A_v^*\}$, v outside some finite set of places S , is a family of semisimple conjugacy classes in $GL_n(\mathbb{C})$. Suppose, in addition, that

$$\prod_{v \notin S} \det(I - A_v^*(Nv)^{-s})^{-1}$$

converges in some right half-plane to an analytic function which has properties similar to those established in the theorem above. Is there then an automorphic cuspidal representation π of G_A such that $A_v(\pi_v) = A_v^*$ for each v outside S ? The answer in general is no. However, an appropriate characterization of “automorphic” families $\{A_v\}$ was given in [JL] for the case $n = 2$ by generalizing the classical converse theorems of Hecke and Weil already described; for $n = 3$, a “converse theorem” is given in [JPS S 1].

Let us briefly discuss some corollaries of the Converse Theorem for GL_2 . As mentioned earlier, Weil’s characterization of the L -functions attached to classical automorphic forms led him to conjecture that the zeta-function

$$\zeta(E, s) = \prod \{ \det[I - A_p^* p^{-s}] \}^{-1}$$

of an elliptic curve E over \mathbb{Q} is really the L -function of an automorphic cusp form of weight 2. The point is that once the zeta-function of such a curve is conjectured to have nice analytic properties, it will follow that the semisimple conjugacy classes $\{A_p^*\}$ in $GL_2(\mathbb{C})$ —which one obtains by counting points on the “reduced curve mod p ”—should comprise an “automorphic” family of conjugacy classes for GL_2 .

On the other hand, Langlands’ local analysis of the constants in the functional equation of Artin’s nonabelian L -functions led Jacquet and Langlands to relate these L -functions as well to “automorphic” L -functions. Indeed, modulo Artin’s conjecture on the entirety of his L -functions, the nonabelian L -functions of degree 2 have been shown to satisfy all the hypotheses of the “Converse theorem” for GL_2 (cf. [Deligne 2]). For arbitrary n , there is the following remarkable “Reciprocity Conjecture”.

CONJECTURE 1 (LANGLANDS). Suppose E is a finite Galois extension of F with Galois group $G = \text{Gal}(E/F)$, and $\sigma: G \rightarrow GL_n(\mathbb{C})$ is an irreducible

representation of G . Then there exists an automorphic cuspidal representation π_σ on GL_n over F such that $L(s, \pi_\sigma) = L(s, \sigma)$.

Note that when $n = 1$ and E over F is abelian, this conjecture reduces to Artin's celebrated "reciprocity law" relating $L(s, \sigma)$ to Hecke's abelian L -series $L(s, \chi)$. Moreover, for arbitrary n , the truth of this conjecture implies the truth of Artin's conjecture on the entirety of his $L(s, \sigma)$, since $L(s, \pi)$ is entire for any (nontrivial) cuspidal representation π of GL_n .

Note also that we have at last succeeded in showing how Langlands' theory (at least conjecturally) subsumes all the classical themes and results discussed in Part II.

B. L -groups and the functoriality of automorphic representations. To proceed deeper into Langlands' program, it is necessary to deal finally with more general groups than GL_n and to introduce the notion of an " L -group". This latter notion is already implicit in the definition of L -functions for GL_n , but needs to be made explicit before these functions can be generalized.

Recall that if $\pi = \otimes \pi_v$ is a representation of $GL_n(\mathbf{A})$, the theory of spherical function assigns to each $v \notin S_\pi$ a well-defined semisimple conjugacy class in $GL_n(\mathbf{C})$. Actually, to be more precise, the theory of spherical functions establishes a bijection between unramified representations π_v and orbits of unramified homomorphisms of the local torus

$$T_v = \left\{ \begin{bmatrix} a_1 & & & \circ \\ & a_2 & & \\ & & \ddots & \\ \circ & & & a_n \end{bmatrix} : a_i \text{ in } F_v^x \right\}.$$

Here "unramified" means that the restriction to

$$T_v(O_v) = \left\{ \begin{bmatrix} \alpha_1 & & & \circ \\ & \ddots & & \\ & & \circ & \\ & & & \alpha_n \end{bmatrix} : \alpha_i \text{ in } O_v^x \right\}$$

is trivial, and orbits are understood taken with respect to the action of the so-called Weyl group (which in this case is just S_n).

Now the structure theory of GL_n is such that these sets of orbits of "quasi-characters" are in natural 1-1 correspondence with semisimple conjugacy classes of the complex group $GL_n(\mathbf{C})$. What the notion of an L -group does is systematize this analysis and extend it to more general reductive groups. Since we are ignoring the definition and structure theory of a general reductive group, we refer the reader to [Cart 2, Springer, Humph or Ti] for details and examples.

Suppose first that G is a "split" such group—an arbitrary connected reductive group with a maximal torus split over F . Guided by the facts just

described for GL_n , Langlands (in [Langlands 2]) constructed a complex reductive Lie group ${}^L G^0$ —the “ L -group of G ”. For $G = GL_n$, of course ${}^L G^0 = GL_n(\mathbb{C})$. Here are some other examples:

G	${}^L G^0$
SL_n	$PGL_n(\mathbb{C})$
PGL_n	$SL_n(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$

In general, the construction of ${}^L G^0$ uses the notions of maximal tori, root systems, etc., concepts originally introduced to classify the complex Lie algebras and their finite-dimensional representations. The result is a bijection between the unramified representations of G_v and the semisimple conjugacy classes of ${}^L G^0$, i.e., orbits of the maximal torus ${}^L T^0$ with respect to the “Weyl group” of $({}^L G^0, {}^L T^0)$; note that if $G = GL_n$, then ${}^L T^0 = T^n$, and the Weyl group is just (isomorphic to) S^n . In any case, given an irreducible unitary representation $\pi = \otimes \pi_v$ of G_A , with π_v unramified for all v outside S_π , there is defined a collection of conjugacy classes $\{A_v\}$ in ${}^L G^0$.

DEFINITION. If r is any finite-dimensional complex analytic representation of ${}^L G^0$, define

$$L_v(s, \pi, r) = \det[I - r(A_v)Nv^{-s}]^{-1}$$

for $v \notin S_\pi$, and

$$L(s, \pi, r) = \prod_{v \notin S_\pi} L_v(s, \pi, r).$$

Here, as before, Nv denotes the cardinality of the finite field O_v/P_v .

These L -functions generalize those already defined for GL_n in Part III. Indeed, if $G = GL_n$, and $r: GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$ is the obvious “standard” representation, then $L(s, \pi, r) = L(s, \pi)$. On the other hand, fixing $G = GL_n$, and letting r vary over all possible representations (and dimensions), we obtain a family of L -functions for GL_n .

CONJECTURE 2'. Suppose π is actually automorphic. Then $L(s, \pi, r)$, initially defined only in some right half-plane, continues meromorphically to \mathbb{C} with a functional equation relating $L(s, \pi, r)$ to $L(1 - s, \tilde{\pi}, r)$.

Unfortunately, there are very few examples of verifications of Conjecture 2'. One example, however, is particularly closely related to the “principle of functoriality” which we shall soon discuss.

Suppose $G = GL_2$ (so ${}^L G^0 = GL_2(\mathbb{C})$), and let r denote the 3-dimensional adjoint representation of $GL_2(\mathbb{C})$ obtained by composing the natural adjoint action of $PGL_2(\mathbb{C})$ (on the 3-dimensional Lie algebra of trace zero 2×2

matrices) with the natural projection map $GL_2(\mathbb{C}) \rightarrow PGL_2(\mathbb{C})$. Thus we have the diagram

$$\begin{array}{ccc} {}^L G^0 = GL_2(\mathbb{C}) & \xrightarrow{r} & GL_3(\mathbb{C}) \\ & \searrow & \nearrow \text{Ad} \\ & & PGL_2(\mathbb{C}) \end{array}$$

In [GeJa] it is shown that Conjecture 2' is true for this G and r . Note, in this case, that

$$L_v(s, \pi, r) = \det[I - r(A_v)Nv^{-s}]^{-1} = \det[I - [A'_v]Nv^{-s}]^{-1}$$

is an Euler factor of degree 3. What is shown in [GeJa], using the Converse Theorem for GL_3 , is that the family of conjugacy classes $\{r(A_v)\} = \{A'_v\}$ in $GL_3(\mathbb{C}) = {}^L(GL_3)^0$ is actually "automorphic", i.e. belongs to an automorphic representation $\Pi = \otimes \Pi_v$ of GL_3 . This result is predicted by, and indeed simply a special realization of the principle embodied in Conjecture 3' below.

Suppose G and G' are split groups, and $\rho: {}^L G^0 \rightarrow {}^L G'^0$ is an analytic homomorphism. If r' is any finite-dimensional representation of ${}^L G'^0$, then $r = r' \circ \rho$ is a finite-dimensional analytic representation of ${}^L G^0$. If π is an automorphic cuspidal representation of G , and $v \notin S_\pi$, let A'_v denote the semisimple conjugacy class in ${}^L G'^0$ which contains $\rho(A_v)$. Then

$$L(s, \pi, r) = \prod_{v \notin S} \det[I - r'(A'_v)Nv^{-s}]^{-1},$$

and the analytic continuation and functional equation of the function *on the right*, i.e., for the family A'_v , would follow from Conjecture 2'. Thus we are led to the following "functoriality principle" of Langlands:

CONJECTURE 3'. Given an analytic homomorphism $\rho: {}^L G^0 \rightarrow {}^L G'^0$, and an automorphic representation $\pi = \otimes \pi_v$ of G , there is an automorphic representation π' of G' such that $S_\pi = S_{\pi'}$, and such that for each $v \notin S_\pi$, A'_v is the conjugacy class in ${}^L G'^0$ which contains $\rho(A_v)$. In particular,

$$L(s, \pi', r') = L(s, \pi, r' \circ \rho)$$

for each finite-dimensional representation r' of ${}^L G'^0$.

In order to minimize the brain strain that results from the unfolding of these conjectures, it is helpful to note that each is actually contained in an appropriately formulated functoriality conjecture. For example, suppose we take $G' = GL_n$ and r' to be the standard representation of $GL_n(\mathbb{C})$. Then $L(s, \pi, r' \circ \rho) = L(s, \pi, r) = L(s, \pi')$, and the functional equation and analytic continuation of $L(s, \pi, r)$ would be assured by Theorem 1 if π' were indeed automorphic. In other words, Conjecture 3' implies Conjecture 2'.

In order to also incorporate Conjecture 1 (Langlands' reciprocity conjecture) it is necessary to slightly reformulate Conjecture 3' by taking into account the fact that, unlike GL_n , not all groups are "split" over F . Without going into details (to which we refer the reader to [Bo]), we collect some fundamental facts below.

Suppose G is an arbitrary connected reductive group defined over F , and K is a (sufficiently large) Galois extension of F . Then one can define a complex reductive Lie group ${}^L G^0$, together with an action of $\text{Gal}(K/F)$ on ${}^L G^0$, such that the resulting semidirect product, ${}^L G = {}^L G^0 \rtimes \text{Gal}(K/F)$, satisfies the following properties:

(i) In case G is “split” over F (in particular, when $G = \text{GL}_n$), ${}^L G$ reduces to a direct product (the action of $\text{Gal}(K/F)$ on ${}^L G^0$ being trivial);

(ii) In general, if v is a prime of F unramified in K , with corresponding Frobenius automorphism Fr_v , there is a 1-1 correspondence between “unramified” representations π_v of $G_v = G(F_v)$ and conjugacy classes $t(\pi_v)$ in ${}^L G$ such that the projection of $t(\pi_v)$ onto $\text{Gal}(K/F)$ is the class of Fr_v ; see [Bo] for more details.

This group ${}^L G$, which plays the same role for arbitrary G as $\text{GL}_n(\mathbb{C})$ plays for the group G , is called the (Galois form of the) L -group of G . By a representation of ${}^L G$ we understand a homomorphism $r: {}^L G \rightarrow \text{GL}_d(\mathbb{C})$ whose restriction to ${}^L G^0$ is complex analytic. By an L -homomorphism of L -groups ${}^L G$ and ${}^L G'$ we understand a continuous homomorphism

$$\begin{array}{ccc} \rho: {}^L G & \rightarrow & {}^L G' \\ & \searrow & \swarrow \\ & \text{Gal}(K/F) & \end{array}$$

which is compatible with the natural projections of each group onto $\text{Gal}(K/F)$ (and whose restriction to ${}^L G^0$ is a complex analytic map of ${}^L G^0$ to ${}^L G'^0$).

In terms of these concepts we can finally formulate the ultimate generalizations of Conjectures 2' and 3'.

CONJECTURE 2. Suppose $\pi = \otimes \pi_v$ is an automorphic representation of G and r is a finite-dimensional representation of ${}^L G$. Then the Euler product

$$L(s, \pi, r) = \prod_{\substack{v \\ \text{unramified}}} \det[I - r(t(\pi_v))Nv^{-s}]^{-1},$$

initially defined in a right half-plane of s , continues meromorphically to all of \mathbb{C} with functional equation relating $L(s, \pi, r)$ to $L(1 - s, \tilde{\pi}, r)$.

CONJECTURE 3. (Functoriality of automorphic forms with respect to the L -group). Suppose G and G' are reductive groups and $\rho: {}^L G \rightarrow {}^L G'$ is an L -homomorphism. Then to each automorphic representation $\pi = \otimes \pi_v$ of G there is an automorphic representation $\pi' = \otimes \pi'_v$ of G' such that for all $v \notin S_\pi$ (i.e. unramified v), $t(\pi'_v)$ is the conjugacy class in ${}^L G'$ which contains $t(\pi_v)$. Moreover, for any finite-dimensional representation r' of ${}^L G'$,

$$L(s, \pi', r') = L(s, \pi, r' \circ \rho)$$

CONCLUDING REMARKS. This last conjecture of Langlands' really does imply all the preceding conjectures discussed heretofore.

First take $G' = \text{GL}_n$ and suppose r is any n -dimensional representation of ${}^L G$ (G arbitrary but fixed). Let $\rho: {}^L G \rightarrow {}^L G'$ be such that the following diagram commutes:

$$\begin{array}{ccc} {}^L G & \xrightarrow{r} & \text{GL}_n(\mathbf{C}) \\ \rho \searrow & & \nearrow \text{St} \\ & & {}^L G' = \text{GL}_n(\mathbf{C}) \times \text{Gal}(K/F) \end{array}$$

(Here $\text{St}: {}^L G \rightarrow \text{GL}_n(\mathbf{C})$ is the standard representation of ${}^L G$, namely projection onto the component ${}^L G^0 = \text{GL}_n(\mathbf{C})$.) Then *assuming* the truth of Conjecture 3, we have a lift of automorphic forms $\pi \rightarrow \pi'$ between G and G' with

$$L(s, \pi, r) = L(s, \pi, \text{St} \circ \rho) = L(s, \pi', \text{St}) = L(s, \pi'),$$

with the last L -function (on GL_n) “nice” by [GoJa]. Thus Conjecture 3 not only implies Conjecture 2, but also reduces the study of generalized L -functions for arbitrary G to the known theory for GL_n !

Now suppose $G = \{e\}$ (the trivial group) and again take $G' = \text{GL}_n$. Then the only possible automorphic representation of G is the trivial one, and an L -homomorphism $\rho_\sigma: {}^L G \rightarrow {}^L G'$ amounts to specifying a representation $\sigma: \text{Gal}(K/F) \rightarrow \text{GL}_n(\mathbf{C})$ such that $\rho_\sigma(1 \times \gamma) = \sigma(\gamma) \times \gamma$ for all $\gamma \in \text{Gal}(K/F)$. Thus Conjecture 3 amounts to the assertion that there is an automorphic representation π_σ of GL_n (associated to the trivial representation of G via $\rho = \rho_\sigma$) such that for all unramified primes v , the projection of $t(\pi_v)$ on $\text{GL}_n(\mathbf{C})$ is just $\sigma(\text{Fr}_v)$. In particular,

$$L(s, \pi_\sigma) = L(s, \sigma),$$

which means Conjecture 3 indeed implies Conjecture 1 (Langlands’ Reciprocity Conjecture).

Note. A careful and detailed exposition of Langlands’ general program is found in [Bo]; our brief sketch of the theory follows [Art].

C. What’s known? Though many specific cases of the functoriality conjecture (Conjecture 3) have been verified, it is far from being solved. The most comprehensive survey of known results (and work in progress) is found in [Bo]. The reader is also referred to [Langlands 3] for a concise, illuminating discussion.

In the paragraphs below, we shall comment briefly only on a small part of the recent work in this area.

1. *Artin’s Conjecture and Langlands’ Reciprocity Conjecture.* Here we follow [Art] quite closely. Suppose K is a Galois extension of F , and $F = \mathbf{Q}$ (for simplicity). We think of K as the splitting field of some monic polynomial $f(x)$ with integer coefficients. For almost all primes p , namely those “unramified” in K , we let Fr_p denote the (conjugacy class of a) Frobenius automorphism in $\text{Gal}(K/\mathbf{Q})$. Recall that the prime p “splits completely” in K (i.e., the ideal it generates in the ring of integers O_K factors into $[K:\mathbf{Q}]$ distinct prime ideals of O_K) if and only if $\text{Fr}_p = \text{Id}$. In terms of the polynomial $f(x)$, this means (in general) that $f(x)$ splits into linear factors “mod p ”.

Let $S(K)$ denote the set of primes p which split completely in K . For example, if $K = \mathbf{Q}(\sqrt{-1})$, then $S(K) = \{p: p \equiv 1 \pmod{4}\}$. In general, it is known that the map $K \rightarrow S(K)$ is an *injective* order reversing map from finite Galois extensions of \mathbf{Q} into subsets of prime numbers. In other words, the set of splitting primes determines K uniquely. Thus it is natural to pose the following:

Problem. What is the image of this map? I.e., what sets of prime numbers are of the form $S(K)$?

A solution to this problem would constitute some kind of “nonabelian class field theory” since we would be able to parametrize all the finite Galois extensions K of \mathbf{Q} by the collections $S(K)$ (which are intrinsic to \mathbf{Q}). In case we restrict attention to *abelian* extensions, a solution is known in terms of congruence conditions like those for the example $K = \mathbf{Q}(\sqrt{-1})$; this is the “abelian class field theory” discussed in Parts I and II. In general, such a neat solution cannot be expected, but any intrinsic characterization of these sets $S(K)$ certainly deserves to be called a reciprocity law.

What light do Langlands’ ideas shed on this fundamental problem?

Let $\bar{\mathbf{Q}}$ denote an algebraic closure of \mathbf{Q} . Given a Galois extension K of \mathbf{Q} as above, there exists a homomorphism $\sigma: \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{GL}_n(\mathbf{C})$ with the property that $\text{Gal}(\bar{\mathbf{Q}}/K)$ is the kernel of σ . Thus we get an *injective* homomorphism $\sigma: \text{Gal}(K/\mathbf{Q}) \rightarrow \text{GL}_n(\mathbf{C})$ to which we can attach the Artin L -function $L(s, \sigma)$ discussed in II.C.2. Moreover, the definitions are such that

$$S(K) = \{p: \sigma(\text{Fr}_p) = I\}.$$

Now consider again Langlands’ Conjecture 1. It asserts that the family $\{\sigma(\text{Fr}_p)\}$ is automorphic, i.e., that there exists an automorphic representation $\pi = \otimes \pi_p$ of GL_n such that for all p outside S_n , $A_p = \sigma(\text{Fr}_p)$. In particular,

$$S(K) = \{p: A_p = I\}.$$

Thus (the truth of) Conjecture 1 reduces the reciprocity problem above to the study of automorphic representations of $\text{GL}_n(\mathbf{A})$. Moreover, this relation is typical of the perspectives which Langlands’ program brings to classical number theory.

The fact that the collections $S(K)$ —which classify Galois extensions of \mathbf{Q} —might be recovered from data obtained analytically from the decomposition of the right regular representation R into irreducibles is not only abstractly satisfying, it also gives us a handle on solving the original problem. Indeed, when $n = 2$, Langlands has already applied the theory of representations to prove Conjecture 1 for a wide class of irreducible representations σ of $\text{Gal}(K/F)$. For a discussion of these matters, see [Ge 3, Ger Lab] and the original sources [Langlands 3, 4]; the most recent results are described in [Tun].

2. *Other examples.* Below we provide a partial list of recent verifications of the functoriality principle; for more complete discussions and references the reader is again referred to [Bo].

(a) *Base-change.* Take E to be a cyclic Galois extension of F of prime degree. Then automorphic representations of GL_2 over F “lift” to automorphic

representations of GL_2 over E corresponding to the following homomorphism of L -groups. If $G = \mathrm{GL}_2$, then the group $G' = \mathrm{Res}_F^E G$ (obtained “by restriction of scalars”) is such that G'_F (the points of G' over F) = $\mathrm{GL}_2(E)$; its L -group is ${}^L G' = \mathrm{GL}_2(\mathbf{C}) \times \cdots \times \mathrm{GL}_2(\mathbf{C}) \rtimes \mathrm{Gal}(E/F)$, the Galois group acting on ${}^L G'^0 = \mathrm{GL}_2(\mathbf{C}) \times \cdots \times \mathrm{GL}_2(\mathbf{C})$ by permuting coordinates. The L -group homomorphism ρ which gives rise to “base change lifting” is then given by simply imbedding ${}^L G^0$ diagonally in ${}^L G'^0$. The resulting “lifting theorems” (due to Saito, Shintani and Langlands) play a fundamental role in Langlands’ proof of his Conjecture 1 for (certain) two-dimensional Galois representations σ .

(b) *Zeta-functions of algebraic varieties.* We have already alluded to Weil’s conjecture asserting that the zeta-function of an elliptic curve is “automorphic”. Similar results have been conjectured more generally for the zeta-functions (counting points mod p) for general algebraic varieties over F . The classical results center around “Eichler-Shimura theory”, and a general theory has been developed by Langlands, Deligne, Milne, Shih, Shimura and others. For an elementary introduction, see [Ge 2]; for a survey of recent developments, see [Bo, Cas], and the references therein.

(c) *Local results.* Although we have not emphasized this fact, nearly all the assertions thus far treated have local counterparts which are part and parcel of the global theory. For example, the local part of Langlands’ Reciprocity Conjecture amounts to a (conjectured) parametrization of the irreducible representations of $G_v = \mathrm{GL}_n(F_v)$ by n -dimensional representations of the Galois group of F_v (such that ε -factors are preserved.) For $n = 2$ this is already a highly nontrivial and very interesting assertion, the truth of which has just recently been verified by P. Kutzko; see [Cart 1] for a complete exposition of the problem. For GL_n over a p -adic field, especially when $p \nmid n$, see [Moy and Hen].

3. *Related questions.* There are other important directions in the theory of automorphic forms (cf. [Mazur Wiles, Gross B and Rib]) which do not as yet fit in neatly with the general Langlands program. Clearly it would be profitable to pursue the connections with these works and other purely diophantine investigations.

D. Methods of proof. Thus far, three general methods have been used to attack automorphic problems such as the functoriality conjecture. We shall merely sketch the barest outline of these methods and some representative examples of their successes. Undoubtedly, totally new methods are called for as well.

(1) *θ -series.* This is perhaps the oldest, most venerable approach, which began with the classical discovery that

$$\theta(z) = \sum_{-\infty}^{\infty} e^{\pi i n^2 z}$$

defines an automorphic form; more generally, as we already remarked in II.B, similar (automorphic) theta-series can be attached to quadratic forms in any number of variables.

The representation-theoretic construction of automorphic forms via theta-series has as its point of departure the fundamental paper [We 2] published in 1964. In that paper, Weil found a proper group-theoretic home for general theta-functions $\theta(g)$, namely the symplectic groups $\mathrm{Sp}_{2n}(\mathbf{A})$ and their two-fold covering groups (the so-called metaplectic groups). Thus Weil was able to reinterpret the extensive earlier works of C. L. Siegel on quadratic forms and to open the way for group representation theory to be used in the construction of (automorphic) theta-series.

According to Weil, the proper generalization of the classical theta-series $\theta(z)$ is a certain *automorphic* representation of $G_{\mathbf{A}} = \mathrm{Sp}_{2n}(\mathbf{A})$, called *Weil's representation*. This representation acts by right translation in a space of generalized theta-functions $\theta(g)$; by analyzing (in particular, decomposing) this representation, a great deal of interesting information about theta-series can be obtained. For example, in order to understand Hecke's construction of theta-series attached to grossencharacters of a quadratic extension of \mathbf{Q} , one simply decomposes (an appropriate tensor product of) Weil's representation of $\mathrm{SL}_2(\mathbf{A}) = \mathrm{Sp}_2(\mathbf{A})$ into irreducible (automorphic) representations; this is what was carried out in [Sha Tan]. For a survey of similar applications of Weil's representation to the construction of automorphic forms, see [Ge 4].

Now what is the connection between these constructions and the functoriality conjecture of Langlands? The best way to answer this question is to bring into play R. Howe's theory of "dual reductive pairs", yet another simple principle of great beauty and consequence.

Suppose G and G' are subgroups of Sp_{2n} which are each others' centralizers, i.e., (G, G') comprises a "dual reductive pair" in the sense of [Ho 1].

The decomposition of Weil's representation restricted to $G \times G'$ should then give a correspondence $\pi \rightarrow \pi'$ attached to some L -group homomorphism ${}^L G \rightarrow {}^L G'$. Indeed, this restriction should decompose as a sum of representations $\pi \otimes \pi'$ with π' an irreducible representation of G' uniquely determined by π (a representation of G); moreover, π' should be automorphic if and only if π is. For a careful description of this "duality correspondence", see [Ho 1]. In practice, one can usually construct the (global) correspondence $\pi \rightarrow \pi'$ directly by using a formula like

$$(1) \quad f(h) \rightarrow \phi_f(g) = \int_{G_F \backslash G_{\mathbf{A}}} \theta(g, h) f(h) dh;$$

here f is an automorphic form on $G_{\mathbf{A}}$ in the space of π , $\theta(g, h)$ is a theta-function on $\mathrm{Sp}_{2n}(\mathbf{A})$ restricted to $G_{\mathbf{A}} \times G'_{\mathbf{A}}$, and ϕ_f is a function on $G'_{\mathbf{A}}$ which generates the automorphic representation π' . (This operator essentially projects the restriction of Weil's representation onto the isotypic component "belonging to the irreducible representation π ".)

EXAMPLES FOR (1). (i) Take $(G, G') \subset \mathrm{Sp}_4$ with G the norm 1 group of a quadratic extension E of \mathbf{Q} , and $G' = \mathrm{SL}_2 = \mathrm{Sp}_2$. Then the corresponding lift $\chi \rightarrow \pi_{\chi}$ defined by (1) generalizes the construction of Hecke's just alluded to. In particular, if E over \mathbf{Q} is *real*, one obtains Maass' construction of nonholomorphic modular forms; cf. [Ge 1] for details.

(ii) Let G (resp. G') denote the unitary group of an isotropic Hermitian space in two (resp. three) variables over a quadratic extension E of F . Then (G, G')

naturally embeds as a dual reductive pair in Sp_{12} , and the corresponding lift $\pi \rightarrow \pi'$ has been analyzed via (1) in [Ge PS 1].

We note that the *exact* relation between the duality correspondence just sketched and the Langlands lifting predicted by Conjecture 3 is not at all transparent, and in fact is a bit delicate; the interested reader is referred to the Introduction of [Ra 1] for more discussion of “Langlands functoriality for the Weil representation”.

(2) *L-functions*. This method has already been discussed at some length in this paper. Its success in constructing automorphic representations (and thereby verifying the functoriality conjecture) is based mainly on the “Converse Theorem”, which asserts that a representation of G_A is automorphic if and only if enough of its L -functions are “nice”. To be sure, since this “Converse Theorem” has been proved in a useful form only for GL_2 and GL_3 , the range of applicability of this method is somewhat limited (see, however, the remarks following the examples below).

EXAMPLES FOR (2). (i) Take E to be a cubic (not necessarily Galois) extension of a number field F , $G = E^x$ (actually $\mathrm{Res}_F^E \mathrm{GL}_1$), and $G' = \mathrm{GL}_3$. Then there is a natural L -homomorphism $\rho: {}^L G \rightarrow {}^L G'$ (corresponding roughly to the “toral” embedding of ${}^L G^0 = C^x \times C^x \times C^x$ into $\mathrm{GL}_3(\mathbb{C})$), and the corresponding lift $\chi \rightarrow \pi_\chi$ between grossencharacters of E^x and automorphic representations of GL_3 is such that $L(s, \pi_\chi) = L(s, \chi)$. The converse theorem works here because $L(s, \chi)$ (and hence $L(s, \pi_\chi)$) is known to be “nice” by Hecke’s theory of abelian L -functions (cf. Parts II.C.1 and II.D.1); analogous arguments for quadratic extensions and GL_2 give a different approach to Example (1)(i).

(ii) If $G = \mathrm{GL}_2$, $G' = \mathrm{GL}_3$, and $\rho: \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_3(\mathbb{C})$ is the adjoint representation described in Example III.B, then the corresponding lift $\pi \rightarrow \Pi$ has already been discussed.

(iii) Take G equal to the unitary group in three variables over E introduced in Example (1)(ii), and $G' = \mathrm{Res}_F^E G$ (so $G'_F = \mathrm{GL}_3(E)$). Then one can attach to each automorphic cuspidal representation π of G_A an L -function (of degree 6 over F) which is meromorphic with functional equation and (by the converse theorem for GL_3 over E) belongs to an automorphic representation π' of $\mathrm{GL}_3(\mathbf{A}_E)$; cf. [Ge PS 1]. The resulting correspondence $\pi \rightarrow \pi'$ then defines a “base change lift” for U_3 .

There are more subtle (and recent) applications of the theory of L -functions, especially to the functoriality principle, which go beyond the confines of the converse theorem. I have in mind mostly the use of L -functions in characterizing the image of the liftings which come from dual reductive pairs, for example the deep work of [Wald] characterizing the image of the so-called Shimura correspondence, and the work of [PS] on the Saito-Kurokawa lifting between PGL_2 and Sp_4 ; this work promises to shed light on a wide variety of examples, in particular, the (as yet undeveloped) theories of base change for the metaplectic group and PGSp_4 .

Equally worthy of mention is the recent work of [JPSS 3] on base change for GL_2 to a cubic *nonnormal* extension E of F . As mentioned in IV.C.2(a), “base-change” was fully developed for GL_2 (cf. [Langlands 4]) for cyclic Galois

extensions of F of prime degree. To prove the theorem for nonnormal E an appeal is made to the theory of L -functions on $GL_n \times GL_m$ as developed in [JPSS 2]; moreover, to date at least, the trace formula proof used for Galois extensions (see below) has not been made to work for nonnormal E .

(3) *The trace formula.* This method is both the newest and “hottest” approach to studying automorphic representations. Without pretending to go to the heart of the matter we shall give a very rough idea of how this method works. Unfortunately, the subject is in such a state of flux and development that even the experts can get confused and frustrated.

What is “the trace formula”? For a given group G , let us consider our friend the right regular representation R (cf. II.D) of $G_{\mathbb{A}}$ acting in $L^2_0(G_F \backslash G_{\mathbb{A}})$. It decomposes as a sum of irreducible (unitary) automorphic representations π of G , each with finite multiplicity m_{π} , i.e.,

$$(2) \quad R = \bigoplus m_{\pi} \pi.$$

Since one goal of the theory of automorphic forms is to understand which π occur in (2), and since an irreducible representation π is determined by its “character”, it is natural to want to compute the character of R . This is what “the trace formula for G ” does. Of course we can’t just take the trace of a unitary representation (it doesn’t exist!), so we have first to integrate the representation against a “nice” compactly supported function f on $G_{\mathbb{A}}$ (getting the operator $R(f) = \int f(g)R(g) dg$ which can be shown to be of trace class). On the one hand, we have

$$(3) \quad \text{tr } R(f) = \sum m_{\pi} \chi_{\pi}(f),$$

where

$$\chi_{\pi}(f) = \text{trace}\{ \int f(g)\pi(g) dg \}.$$

On the other hand, using the explicit form of $R(f)$, as an integral operator in $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$, we get a second (more complicated) expression for the trace which makes no reference to the decomposition (2); instead it involves expressions intrinsic to the geometry of the group, for example “orbital integrals” of f over “rational” conjugacy classes in $G_{\mathbb{A}}$, etc.

The idea is that by carefully examining this “second form of the trace formula” one should be able to conclude something about the expression (3), i.e., about the automorphic representations π of $G_{\mathbb{A}}$. In practice, however, this turns out to be nearly impossible, i.e., it is difficult to show that a given representation π of $G_{\mathbb{A}}$ occurs in (2) by just analyzing an explicit formula for $\text{tr } R(f)$. What *does* seem to work, however, is to *compare* the (second forms of the) trace formula for two *different* groups G and G' and then conclude that if π occurs in $(R, G_{\mathbb{A}})$ then π' will occur in $(R', G'_{\mathbb{A}})$.

For example, suppose G is the multiplicative group of a division quaternion algebra and $G' = GL_2$. In §16 of [JL] the trace formulas for these groups are compared, and the conclusion is that there is a correspondence $\pi \rightarrow \pi'$ between the (greater than one-dimensional) automorphic representations of G and a certain subset of the automorphic π' on $GL_2(\mathbb{A})$. In fact, this correspondence is

consistent with the *identity* homomorphism between the L -groups of G and G' ; in particular, $L(s, \pi) = L(s, \pi')$.

As already suggested, the prototype example of this approach to the functoriality principle is the base-change lifting introduced in Example IV.C.2(a). Other examples include:

(i) A new treatment of the “Adjoint” lifting $GL_2 \rightarrow GL_3$ due to Flicker [Flick 1]; here, as in all such applications of the trace formula, one can characterize the image of the lifting as well;

(ii) A different proof of base change and lifting for the unitary group U_3 ; cf. [Flick 2].

Warning. Most applications of the trace formula involve pairs of groups G, G' whose conjugacy classes are not so easily compared as suggested in the examples above; this leads to the thorny notions of “instability” for the trace formula, “ L -indistinguishability” and the like. Though these notions arose initially as impediments to a direct application of the trace formula, Langlands figured out how to turn them into powerful weapons for *proving* his Functoriality Conjecture. Indeed, these suggestions have already been followed in some of the works just mentioned; for a comprehensive but difficult introduction to this increasingly active research domain, see [Langlands 6, Shel 5 and Flick 1, 2].

CONCLUDING REMARKS. It is a happy circumstance that the three methods of proof just sketched complement each other remarkably well. For example, sometimes a deep theorem can be proved *only* by using a mixture of two or three of these approaches. An example of this is Langlands’ Reciprocity Conjecture for “tetrahedral” two-dimensional representations of $\text{Gal}(K/F)$; cf. [Langlands 3 or Ge 3] for a leisurely discussion. On the other hand, sometimes a result can be proved by using *any* one of these approaches, but each method affords its own particular advantages. An example of this is the lifting of automorphic representations from a division quaternion algebra to GL_2 ; proofs using L -functions or the trace formula are found in [JL], and a direct proof using theta-series is the subject matter of [Shimizu]. A slightly different kind of example is the correspondence between automorphic representations of the metaplectic group and GL_2 (Shimura’s correspondence); cf. [Shimura, Ge PS 1, Flick 3 and Wald]. Thus, as already suggested by our discussion of base change, no one of these methods has a monopoly on proving interesting theorems.

E. A few last words. It should be clear by now that the strength of Langlands’ program lies as much (or more) in suggesting new problems as well as in resolving old ones.

Given two groups G and G' , and an L -homomorphism between them, what is the relation between the automorphic representations of G and G' ?

Given a number-theoretically defined family of conjugacy classes in some complex group like $GL_n(\mathbb{C})$, what is the “automorphic nature” of this collection?

Questions like these will undoubtedly keep mathematicians busy for a long time to come.

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Code for Superscripts

- a = General reference
- b = Langlands program
- c = Representation theory (real or p -adic groups)
- d = Automorphic L -functions (local or global theory)
- e = θ -series
- f = Trace formula
- g = Zeta-functions of varieties
- h = L -indistinguishability
- i = Local global principle
- j = Related developments in number theory

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