solutions as "equilibrium" solutions, and by IJ's studied refusal to use "that" as a relative pronoun, a refusal that could make H. W. Fowler undergo the usual rotational instability in his grave.)

Each of these books offers an effective entrée into a lively area of research and a helpful guide for those who wish to apply the theory. Each book would nicely complement the standard texts used in beginning graduate courses in ordinary differential equations.

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Interpolation of linear operators, by S. G. Kreĭn, Ju. I. Petunin, and E. M. Semenov, Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence, R.I., 1982, xii + 375 pp., $78.00. ISBN 0-6869-1993-9

> The pair have reached that fearful chasm,
> How tempting to bestride!
> For lordly Wharf is there pent in
> With rocks on either side.
>
> WILLIAM WORDSWORTH

Such temptation is the essence of interpolation theory. For example, if an operator is bounded on the Lebesgue space L^1 and on L^∞, then intuition demands that it be bounded also on L^p for $1 < p < \infty$. Similarly, if an operator is bounded on the space C of continuous functions and on the space C^2 of twice-continuously-differentiable functions, one hopes that it will be bounded also on C^1. Hope and glory are not to be confused, however, especially in mathematics, and numerous efforts over the past half-century to resolve such problems have led to the development of a rich and interesting theory of interpolation of operators.

The seminal work is an interpolation theorem for L^p-spaces established by Marcel Riesz in 1926. Operators that are bounded from L^p to L^q are said to be of *strong type* (p, q). If $1 \leqslant p_0, q_0, p_1, q_1 \leqslant \infty$ and if $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$ $(0 \leqslant \theta \leqslant 1)$, then Riesz's theorem asserts that every linear operator of strong types (p_0, q_0) and (p_1, q_1) is also of strong type (p_θ, q_θ).

The operator norm M_θ is a logarithmically convex function of θ, that is, $M_\theta \leqslant M_0^{1-\theta} M_1^\theta$, at least under certain restrictions on the parameters. These restrictions can be removed completely, however, for L^p-spaces with *complex* scalars, as was demonstrated by O. V. Thorin in 1939. Thorin's proof makes clever use of the Hadamard three-lines theorem from complex function theory. Later referred to by J. E. Littlewood as "the most impudent idea in analysis", the proof has become standard in the literature, where the result is usually known as the *Riesz-Thorin convexity theorem*.

The year 1939 saw another major advance in the theory, this one due to J. Marcinkiewicz. The strong-type hypotheses required in the Riesz-Thorin theorem are too restrictive for many potential applications. For example, in 1927, M. Riesz had established the L^p-boundedness $(1 < p < \infty)$ of the Hilbert transform H (so H is of strong type (p, p) for $1 < p < \infty$) but this cannot be deduced from the Riesz-Thorin theorem because H fails to be of strong type $(1, 1)$ (or (∞, ∞)). There is, however, a weaker result of A. N. Kolmogorov (1923) to the effect that H maps L^1 into a somewhat larger space, which is now known as *weak-*L^1. In other words, although H is not of strong type $(1, 1)$, it does have a property which might appropriately be referred to as *weak type* $(1, 1)$.

Enlarging this concept to include values of p other than $p = 1$, Marcinkiewicz proceeded to show that the strong-type conclusions of the Riesz-Thorin theorem remain intact even when the strong-type hypotheses of that theorem are replaced by the corresponding weak-type conditions. Some additional restrictions on the parameters are needed but we need not specify them here. It is important to point out that Marcinkiewicz's proof, in contrast to Thorin's, is distinctly *real*-variable in character, and depends on an adroitly chosen decomposition, by truncation, of a function into its "large" and "small" parts.

Following Marcinkiewicz's death, in tragic circumstances in the early days of the Second World War, the theorem lay dormant for a number of years until his former mentor and collaborator, A. Zygmund, published a much expanded version in 1956, together with proof and many additional applications. With subsequent refinements due to many mathematicians, the Marcinkiewicz theorem has been firmly established as one of the cornerstones of modern harmonic analysis. We mention, in particular, the work of A. P. Calderón which, building on earlier results of E. M. Stein and G. Weiss, provides a natural formulation of the Marcinkiewicz interpolation theorem in terms of the two-parameter family of Lorentz spaces L^{pq}. This in turn leads to an extensive theory of interpolation in rearrangement-invariant Banach function spaces.

There are, of course, many other types of spaces for which interpolation theorems are desirable. The various kinds of spaces of smooth (or analytic) functions—Lipschitz spaces, Sobolev spaces, Besov spaces, Hardy spaces, etc. —are natural candidates, as indeed are any families of spaces defined in terms of one or more parameters. Consolidation of existing results was thus a natural next step, and by the late fifties there began to emerge a unified theory of interpolation in Banach spaces.

Central to the theory is the concept of an *interpolation method*. From each couple (X_0, X_1) of Banach spaces, an interpolation method constructs a family of Banach spaces, say $(X_0, X_1)_\theta$ $(0 < \theta < 1)$, for which the interpolation property *automatically* holds. That is to say, if (X_0, X_1) and (Y_0, Y_1) are any two such couples, then every linear operator which is bounded from X_0 to Y_0 and from X_1 to Y_1 is to have the property that it be bounded also from $(X_0, X_1)_\theta$ to $(Y_0, Y_1)_\theta$.

Two such methods are known and widely used. The first, the so-called *real method* (because of its basis in the "splitting" technique used in the proof of the Marcinkiewicz interpolation theorem), was devised by E. Gagliardo in 1959 and subsequently refined into more useable form by J.-L. Lions and J. Peetre. The second, based on Thorin's proof of the Riesz-Thorin theorem, was developed independently by A. P. Calderón, S. G. Kreĭn, and J.-L. Lions around 1960 and is referred to as the *complex method*. A third method, the *Riesz method*, has been developed very recently by J. Peetre and is based on Riesz's original, rather mysterious, proof of the original interpolation theorem. The implications of this new method are not yet clear and much remains to be done. The complex method has also seen radical improvement recently, with important new applications, largely due to the efforts of R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher, and G. Weiss.

The methods begin with a couple (X_0, X_1) of Banach spaces, that is, a pair of Banach spaces each of which is embedded in a suitable larger space (so that the sum $X_0 + X_1$ and intersection $X_0 \cap X_1$ make sense). In the complex method, the space $(X_0, X_1)_\theta$ is constituted by the values $F(\theta)$ as F varies over the set of $(X_0 + X_1)$-valued analytic functions in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ that take values in X_0 on the left-hand boundary $\{it\colon -\infty < t < \infty\}$ and values in X_1 on the right-hand boundary $\{1 + it\colon -\infty < t < \infty\}$. In the real method, which is most-often exhibited as a two-parameter method, the space $(X_0, X_1)_{\theta,q}$ $(0 < \theta < 1,\ 1 \leqslant q \leqslant \infty)$ is determined by imposing appropriate growth conditions on the *Peetre K-functional*

$$K(f, t) = \inf_{f=f_0+f_1} \left(\|f_0\|_{X_0} + t\|f_1\|_{X_1}\right) \qquad (0 < t < \infty)$$

of elements $f \in X_0 + X_1$. Thus $f \in (X_0, X_1)_{\theta,q}$ if

$$\|f\|_{\theta,q} = \left(\int_0^\infty \left[t^{-\theta} K(f, t) \right]^q \frac{dt}{t} \right)^{1/q}$$

is finite.

The interpolation property is not difficult to prove for either the real or the complex method. The real difficulty arises in trying to identify in concrete terms the interpolation spaces so-constructed by these methods. In the real method, for example, this amounts to identifying the K-functional, but this has been done in many important cases. Thus, for the pair (L^1, L^∞), the K-functional is $\int_0^t f^*(s)\,ds$, where f^* is the decreasing rearrangement of f. The real interpolation spaces are then easily seen to be the Lorentz spaces L^{pq} so that the abstract theory reproduces in this case the classical Marcinkiewicz result. The K-functional for the pair (C, C^k) is essentially the kth order modulus of continuity so that Lipschitz spaces occur as the real interpolation spaces. Similarly, for the pairs (H^1, L^∞) and (L^1, BMO), the K-functionals are expressible in terms of the grand maximal function and the Fefferman-Stein sharp function, and so on.

The theory of interpolation methods has become an indispensable tool in areas such as partial differential equations, approximation theory, and harmonic analysis. The literature is extensive and includes several good books on different aspects of the theory. No single work can hope to cover it all and in the book under review the authors have concentrated on providing a solid introduction to the subject. Applications are confined to the last chapter on spaces of smooth functions (contributed by S. G. Kreĭn and absent, for technical reasons, in the original Soviet edition).

The functional-analytic foundations of the subject are laid in the first chapter. Much of this material was developed originally by N. Aronszajn and E. Gagliardo in a more general setting. The authors have done an excellent job of distilling this formidable body of material into their concise and readable account.

The second chapter returns to the origins of the subject and develops the theory of interpolation on spaces of measurable functions. It contains a comprehensive treatment of rearrangements of functions which, although omitted in some texts, would seem to us to be essential to any proper understanding of the abstract theory. The main interpolation theorems for rearrangement-invariant spaces are developed from an analysis of measure-preserving transformations (by contrast, the original work of Calderón proceeded by means of a limiting argument from a result of Hardy, Littlewood, and Pólya on doubly stochastic matrices), which makes for a cohesive account and one which could well become a standard reference for workers in the area.

The remaining chapters describe in detail the real and complex methods and their relationships with the associated idea of a "scale" of spaces. Interesting applications are interspersed throughout the book, although, as we have mentioned earlier, the main thrust is to provide a thorough introduction to the subject of interpolation itself. The translation is lucid, professionally done, and reads well. All in all, the book is a welcome addition to the literature. Wordsworth, we are sure, would have approved.

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Generalized solutions of Hamilton-Jacobi equations, by P. L. Lions, Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program, Boston, 1982, 317 pp., $24.95. ISBN 0-2730-8556-5

Introduction. The Hamilton-Jacobi equation is probably known to most engineers and physicists as a partial differential equation which pops up in the study of (Lagrangian or Hamiltonian) mechanics, yielding solutions of a system of ordinary differential equations, as its characteristics, after a variational procedure is used. It is also known, again through its relation to the calculus of variations, to people studying control theory, differential games, or other optimization problems, although it is sometimes referred to as the "Bellman equation" in these contexts.

The last thirty years has seen the rise of a new interest in the Hamilton-Jacobi equation. With the rise of computers and new numerical techniques, the failure of classical smooth solutions to describe physical situations except in limited (local) domains, and the needs of mathematical modeling, aerospace engineering, and other applications to have solutions described everywhere, many mathematicians have become interested in global solutions (whatever that means). As nearly the most general first order partial differential equation, and as an equation for which global results were possible, the Hamilton-Jacobi equation became a natural target for mathematicians studying global solutions.

In order to clarify the object of interest a little better, let us define the Hamilton-Jacobi equation. In its most familiar classical form, the Hamilton-Jacobi equation is

$$\partial u/\partial t + H(t, x, Du) = 0,$$

where H is a given function, called the Hamiltonian, x is in R^n, and Du denotes the gradient of the solution, u, with respect to x. Here t is a single variable (usually called "time"). The separation of the distinguished variable "t" from the gradient, Du, in H, makes the Hamilton-Jacobi equation much easier to handle than the general first order equation. The Cauchy (or initial value) problem is always noncharacteristic, thus amenable to solution. This same separation of t also makes the Hamilton-Jacobi equation essentially an evolution equation, thus allows a mass of evolution equation techniques to be brought to bear.

The Hamilton-Jacobi equation, as defined by Professor Lions, is

$$H(x, u, Du) = 0,$$