

## EXPLICIT RELAXATION OF A VARIATIONAL PROBLEM IN OPTIMAL DESIGN<sup>1</sup>

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Our goal is to construct a quasiconvex function  $\Phi$  such that

$$(1) \quad \inf_{u|_{\partial\Omega}=F} \int_{\Omega} (1_{\text{supp } \nabla u} + |\nabla u|^2) dx = \inf_{u|_{\partial\Omega}=F} \int_{\Omega} \Phi(\nabla u) dx$$

for vector-valued functions  $u$  on Lipschitz domains  $\Omega \subset \mathbf{R}^2$ . The right side of (1) is the relaxation of the left, cf. [1]. Each infimum is over  $u \in H^1(\Omega; \mathbf{R}^N)$ ,  $1_{\text{supp } \nabla u}$  denotes the characteristic function of the support of  $\nabla u$ , and  $|\nabla u|^2 = \sum (\partial u^i / \partial x_j)^2$ .

The left side of (1) is a problem of optimal design: it minimizes  $\text{Area}(\Omega \setminus S) + \int_{\Omega} |\nabla u_S|^2 dx$ , among all sets  $S \subset \Omega$ , where  $u_S$  solves the variational problem

$$(2) \quad \inf \left\{ \int_{\Omega} |\nabla u|^2 dx : u|_{\partial\Omega} = F, \nabla u = 0 \text{ on } S \right\}.$$

An application will be described below.

For some choices of  $\Omega$  and  $F$ , this optimal design problem has no solution; in other words, the infimum on the left side of (1) may not be attained. The nonexistence of solutions to related problems has been noted by several authors; see [4] and the references given there. Here, it arises because the function

$$(3) \quad G(\nabla u) = \begin{cases} 1 + |\nabla u|^2, & \nabla u \neq 0, \\ 0, & \nabla u = 0, \end{cases}$$

is not quasiconvex, so the left side of (1) is not lower semicontinuous under weak  $H^1$  convergence. A minimizing sequence  $\{u_n\}$  may be highly oscillatory, and  $S_n = \{\nabla u_n = 0\}$  may develop increasingly complicated microstructure.

The relaxed problem, on the right, is lower semicontinuous, hence the infimum is attained. In fact, its solutions are precisely the weak limits of minimizing sequences for the left side. The introduction of such a relaxed problem is a standard way of dealing with nonexistence. The method has its roots in the work of L. C. Young and E. J. McShane, and contributions have been made by Morrey, Ball, Ekeland, Temam, and Dacorogna, among others; see [2] for further discussion and references.

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For the scalar case  $N = 1$ , the methods of Ekeland and Temam [3] may be used to show that  $\Phi$  is the largest convex function below  $G$ ,

$$\Phi(\nabla u) = \begin{cases} 1 + |\nabla u|^2, & |\nabla u| \geq 1, \\ 2|\nabla u|, & |\nabla u| \leq 1. \end{cases}$$

In the vector case  $N > 1$ , however, this convexification of  $G$  would make the right side of (1) too small. Instead, one must use  $\Phi = QG$ , the quasiconvexification of  $G$ , defined by

$$(4) \quad QG(E) = \inf_{\zeta \in C_0^\infty(U; \mathbf{R}^N)} \int_U G(E + \nabla \zeta) dx$$

for any  $2 \times N$  matrix  $E$ , where  $U$  is the unit square in  $\mathbf{R}^2$ . Dacorogna has proved that  $QG$  is quasiconvex, and that the analogue of (1) holds, for any continuous integrand  $G$  satisfying a mild growth condition [1].

Unfortunately, quasiconvexifications are hard to compute. So far, all examples have involved ordinary convexification in an essential way. An underlying difficulty is the lack of an algebraic condition for quasiconvexity. Morrey and Ball gave a sufficient condition, called polyconvexity (for  $u: \mathbf{R}^2 \rightarrow \mathbf{R}^N$ ,  $G(\nabla u)$  is polyconvex if it is a convex function of  $\nabla u$  and its  $2 \times 2$  minors). Hadamard gave a necessary condition, namely rank-one convexity (also called ellipticity, or the Legendre-Hadamard condition). But polyconvexity is not necessary; the sufficiency of rank-one convexity is open; and the condition  $QG = G$  is neither algebraic nor easy to work with.

We have computed the quasiconvexification of (3), for  $u: \mathbf{R}^2 \rightarrow \mathbf{R}^N$ , using the methods of homogenization.

**THEOREM.** *For  $N > 1$ , the quasiconvexification of (3) is*

$$(5) \quad \Phi(\nabla u) = \begin{cases} 1 + |\nabla u|^2 & \text{for } |\nabla u|^2 + 2D \geq 1, \\ 2(|\nabla u|^2 + 2D)^{1/2} - 2D & \text{for } |\nabla u|^2 + 2D \leq 1, \end{cases}$$

where

$$D^2 = \sum_{1 \leq i < j \leq N} \left( \frac{\partial u^i}{\partial x_1} \frac{\partial u^j}{\partial x_2} - \frac{\partial u^j}{\partial x_1} \frac{\partial u^i}{\partial x_2} \right)^2.$$

This  $\Phi$  is polyconvex, and (1) holds whenever  $\Omega$  is a Lipschitz domain and  $F$  is the boundary value of an  $H^1$  function.

We explain how one arrives at (5). Given a subset  $S \subset \Omega$  and a real number  $0 < \delta < 1$ , let  $w_{S,\delta} \in H^1(\Omega; \mathbf{R}^N)$  solve

$$(6) \quad \operatorname{div}(a_S \nabla w_{S,\delta}) = 0 \quad \text{in } \Omega, \quad a_S \nabla_\nu w_{S,\delta} = f \quad \text{on } \partial\Omega,$$

where  $f$  is the derivative of  $F$  along  $\partial\Omega$ , and

$$a_S(x) = \begin{cases} \delta, & x \in S, \\ 1, & x \in \Omega \setminus S. \end{cases}$$

The dual variational principle for (6) gives  $\int_\Omega a_S |\nabla w_{S,\delta}|^2 dx$  as the infimum of a problem involving  $N$  divergence-free vector fields. If  $\Omega$  is simply connected,

we represent these vector fields using stream functions  $u$ , to arrive at

$$(7) \quad \int_{\Omega} a_S |\nabla w_{S,\delta}|^2 dx = \inf_{u|_{\partial\Omega}=F} \int_{\Omega} a_S^{-1} |\nabla u|^2 dx.$$

When  $\delta = 0$ , (6) becomes Laplace's equation on  $\Omega \setminus S$  with a homogeneous Neumann condition on  $\partial S$ , and the right side of (7) coincides with (2). In particular, the variational problem

$$(8) \quad \inf_{S \subset \Omega} \text{Area}(\Omega \setminus S) + \int_{\Omega} a_S |\nabla w_{S,\delta}|^2 dx$$

is equivalent when  $\delta = 0$  to the left side of (1).

As  $S$  varies with  $\delta > 0$  fixed, the limits of the solutions of (6) may satisfy new equations

$$(9) \quad \text{div}(a^* \nabla w) = 0 \quad \text{in } \Omega, \quad a^* \nabla_{\nu} w = f \quad \text{on } \partial\Omega.$$

These constitute the  $G$ -closure of (6), and they correspond to composite materials obtained by mixing the original two. Lurie and Cherkhaev [5] and Tartar and Murat [6] have independently determined the set  $A_{\delta}(\rho)$  of matrices  $a^*$  attainable in (9) "with volume fraction  $\rho$ ", i.e. by a sequence  $\{S_n\}$  with weak limit  $\lim_{n \rightarrow \infty} 1_{\Omega \setminus S_n} = \rho$ . By virtue of (7), (8) leads to

$$(10) \quad \inf_{u|_{\partial\Omega}=F} \inf_{\substack{0 \leq \rho(x) \leq 1 \\ a(x) \in A_{\delta}(\rho(x))}} \int_{\Omega} \left[ \rho + \sum_{j=1}^N \langle a^{-1} \nabla u^j, \nabla u^j \rangle \right] dx.$$

We computed  $\Phi$  by passing to the limit  $\delta \rightarrow 0$  in (10), and evaluating the second infimum:

$$\Phi(\nabla u) = \inf_{\substack{0 \leq \rho \leq 1 \\ a \in A_0(\rho)}} \rho + \sum_{j=1}^N \langle a^{-1} \nabla u^j, \nabla u^j \rangle,$$

where  $A_0(\rho) = \lim_{\delta \rightarrow 0} A_{\delta}(\rho)$ . The proof that (1) holds for this choice of  $\Phi$  combines the tools of [3] with the constructions of optimal composites given by [5 or 6].

We give an application, only slightly far-fetched, of the optimal design problem implicit in (1). Consider a simply connected domain  $\Omega \subset \mathbf{R}^2$  coated with silver, and a family of current loads  $f^j$ ,  $1 \leq j \leq N$ , to be imposed at  $\partial\Omega$ . The voltage produced by  $f^j$  solves  $\Delta w^j = 0$  in  $\Omega$  with  $\nabla_{\nu} w^j = f^j$  at  $\partial\Omega$ , and  $c_j = \int_{\Omega} |\nabla w^j|^2 dx$  is the rate at which energy is dissipated to heat, neglecting factors involving units. The design problem is to remove as much silver as possible, leaving behind a perfect insulator, with the constraint that the rate of energy dissipation under load  $f^j$  must not exceed a given constant  $C_j > c_j$ ,  $1 \leq j \leq N$ .

If the silver is removed from a set  $S$ , the new vector of voltages  $(w^1, \dots, w^N)$  is just  $w_{S,0}$ , the solution of (6) with  $\delta = 0$ , and the rate of energy dissipation is given by (7). Hence our design problem is

$$(11) \quad \inf_{S \subset \Omega} \sup_{\lambda_j \geq 0} J(S, \lambda),$$

where

$$J(S, \lambda) = \text{Area}(\Omega \setminus S) + \sum_{j=1}^N \lambda_j \left( \int_{\Omega \setminus S} |\nabla w_{S,0}^j|^2 dx - C_j \right).$$

On the other hand, the theorem shows that

$$(12) \quad \inf_{S \subset \Omega} J(S, \lambda) = \inf_{u|_{\partial\Omega} = F_\lambda} \int_{\Omega} \Phi(\nabla u) dx - \sum_{j=1}^N \lambda_j C_j$$

for each  $\lambda$ , where  $F_\lambda^j$  is the integral of  $\sqrt{\lambda_j f^j}$  along  $\partial\Omega$ . We conjecture that (11) is obtained by maximizing (12) with respect to  $\lambda_j \geq 0$ . Even without such a minimax result, each solution of (12) determines a solution of (11) for some  $C'_j = C'_j(\lambda)$ .

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