Although any two dense linearly ordered sets are indistinguishable by first order sentences and only partly distinguishable by monadic properties (Gurevich, Magidor and Shelah), some resolution is possible if the perspective is slightly changed and automorphism groups are considered. For example, there is a first order sentence of the language of group theory that is satisfied by the automorphism group of a 1-homogeneous linearly ordered set if and only if the linearly ordered set is the real line (Gurevich, Holland, Jambu-Giraudet and Glass)—and this does not depend on set theory, nor do similar results.

Like this review, the book has a decidedly model-theoretic slant and relies heavily on the Ehrenfeucht-Fraïssé game technique, which I found helpful. Anyone who has taught a course in model theory knows only too well how difficult it is to find theories in which students can "get their hands dirty" and perform detailed computations. Linear orderings provide one of the few exceptions so it is very nice to see that Rosenstein has peppered his pages with calculations. This is further brought out by many exercises—4.4.4.(3) is most challenging; it contradicts 4.4.4.(5) and is one of the few misprints I found (the others are obvious). The book has clearly been written with the graduate student in mind; I have never read a mathematics book before where the author has so obviously chosen his words carefully to ensure that the reader is in the right frame of mind when he or she comes across each new concept. It can certainly be used for a graduate course in infinite combinatorics or in model theory; in the latter case, it might be used as supplementary reading for such a course, though since all the standard model theory is introduced thoroughly, with proofs, before linear orderings are considered specifically, this is not necessary. However, there is such a mine of information in the book that it will be useful for researchers in these disciplines as well as other mathematicians and is a must for libraries. Because of the price, I fear that they and reviewers will be the only owners! Its appearance now that there is a need to study unstable theories is most propitious. The only complaints I have are that Rosenstein makes no attempt in his book, even in his introduction, to relate his subject to other areas of mathematics—something I have tried to right in this perhaps lopsided review—nor, where possible, to deal with partial orderings when little or no extra work is needed. However, any complaints are most churlish since I learnt a lot from the book, have great admiration for the author's own central work and influential role in the subject, and found this a superbly written book.

A. M. W. GLASS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

0273-0979/83 $1.00 + $.25 per page

Hankel operators on Hilbert space, by S. C. Power, Research Notes in Mathematics, No. 64, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1982, 87 pp., $13.95. ISBN 0-273-08518-2

Hankel operators have been studied on and off for years, because they arise in a variety of problems of complex analysis and operator theory. At their introduction to modern analysts (by Hartman and Wintner [**12**]), it may have seemed as though their theory might progress with that of the somewhat similar looking Toeplitz operators. But this was not to be. Toeplitz operators yielded to an onslaught by the analysts of the fifties, sixties and seventies, while it is only now, more than thirty years after the Hartman and Wintner paper, that real progress on Hankel operators can be reported.

1. Hankel operators. For any square summable sequence $\{a_n\}$, $n = 0, 1, \ldots$, the matrix

$$S = \begin{pmatrix} a_0\, a_1\, a_2 \cdots \\ a_1\, a_2\, a_3 \cdots \\ a_2\, a_3\, a_4 \cdots \\ \cdots \end{pmatrix}$$

defines a Hankel operator on the space l_+^2 of sequences $(x_0, x_1, \ldots)$ with $\Sigma |x_n|^2 < \infty$. A useful equivalent definition of S can be given on H^2, the space of analytic functions $\Sigma x_n z^n$, with $\Sigma |x_n|^2 < \infty$. In fact, S is the matrix of the operator S_φ on H^2 defined by $S_\varphi f = PJ\varphi f$, where P is the projection of L^2 (of the unit circle $|z| = 1$) on H^2, $Jg(e^{it}) = g(e^{-it})$, and φ is any function with the sequence $\{a_n\}$ for nonpositive Fourier coefficients:

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) e^{int}\, dt, \qquad n = 0, 1, 2, \ldots.$$

Note the ambiguity in the choice of φ such that $S = S_\varphi$; this is the crux of several basic theorems about S, as we shall see.

Actually, the above definitions of S and S_φ are not quite right without restricting to an appropriate domain (the polynomials in H^2, for example) or investigating when S or S_φ is bounded. The boundedness question for Hankel operators was solved by Nehari [**16**], who proved that S is bounded if and only if $S = S_\varphi$, for some bounded φ, in which case

$$\|S_\varphi\| = \operatorname{dist}(\varphi, zH^\infty). \tag{1}$$

This theorem was followed by Hartman's Theorem [**11**]: S is compact if and only if $S = S_\varphi$ for some continuous φ; and by characterizations of the s-numbers (the eigenvalues of $(SS^*)^{1/2}$). The latter characterizations, analogous to (1), were first obtained, for selfadjoint S, by the reviewer [**6**], then independently reproved, for general S, by Adamjan, Arov and Krein [**1, 2**].

Actually, the seeds for the above theorems were planted much earlier for *finite* S ($a_n = 0$, for $n \geqslant n_0$), when Carathéodory and Fejér [**5**] proved (1) and Takagi [**24**] found the s-numbers, in connection with some problems of complex interpolation.

2. Relationships with Toeplitz operators. Other problems concerning Hankel operators were studied during the period of rediscovery and generalization of these early results, some motivated by relationships with the more tractable class of Toeplitz operators.

The definition of a Toeplitz operator T_φ: $H^2 \to H^2$ is $T_\varphi f = P\varphi f$, and the simplest relation between T_φ and S_φ is obtained by taking apart the matrix of the operator L_φ of multiplication by φ on L^2. Relative to the decomposition $L^2 = (H^2)^\perp \oplus H^2$, L_φ has the matrix

$$L_\varphi = \begin{pmatrix} J^*T_{zJ\varphi}J & \bar{z}JS_{z\varphi} \\ S_{J\varphi}J & T_\varphi \end{pmatrix}.$$

Compactness of $S_{z\varphi}$ and $S_{J\varphi}$ (true whenever φ is continuous, by Hartman's Theorem) is thus seen to reduce the question of membership in the Fredholm class for the operator T_φ to the same question for the much simpler operator L_φ (Krein [**14**]).

A somewhat different relationship between T_φ and S_φ was given by Putnam [**21**], who showed that if φ is an even function then the matrix of T_φ differs from the matrix of multiplication by φ on $L^2(0, \pi)$ by the matrix of $S_{z^2\tilde{\varphi}}$. For φ real valued, the corresponding selfadjoint perturbation problem led Rosenblum [**23**] to pose the problem of determining when S_φ is trace class. This problem was further studied by Howland [**13**] and the reviewer [**7**].

A third relation is

$$S^*_{z\tilde{\varphi}}S_{z\psi} = T_{\varphi\psi} - T_\varphi T_\psi, \qquad \tilde{\varphi}(z) = \varphi(\bar{z}), \tag{3}$$

which suggests the problem: when is $S_f^*S_g$ compact? By (3), this would have obvious implications for the algebra generated by the Toeplitz operators, modulo the ideal of compact operators; Coburn [**8**], Gohberg and Krupnik [**10**].

3. The latest results. Although Hankel operators had appeared with some frequency there were, by the late sixties, very few theorems about the spectrum of S which were applicable in special cases. In fact, aside from results on Hilbert matrices ($a_n = 1/(n+1)$, Magnus [**15**]; and $a_n = 1/(n+\lambda)$, $0 < \lambda < 1$; Rosenblum [**22**]), essentially no examples were known where the spectrum, $\sigma(S_\varphi)$, or even the essential spectrum, $\sigma_e(S_\varphi)$, could be computed.

This situation was to a large part rectified by the author's appearance upon the scene. Indeed, through a series of papers [**18–20**], Power obtained the essential spectrum of S_φ, for a number of classes of φ, including piecewise continuous and piecewise "slowly oscillating". Add the fact that the trace class Hankel operator problem has recently been solved by Peller [**17**] and Coifman and Rochberg [**9**], and conditions for compactness in (3) have been given by Axler, Chang and Sarason [**4**] and Volberg [**25**], and the time, as well as the author, is right for these notes.

As though this were the time for finally settling questions about Hankel operators, Axler, Berg, Jewell and Shields [**3**] have recently shown that the distance of an L^∞ function to $H^\infty + C$ is attained. By the characterization of the "essential norm" of S in [**6**, **1**, and **2**], this can be translated to a statement about Hankel operators.

4. About the notes. An exposition largely of the above topics, the notes of Power appear as a series of lectures, with the ordering of topics dictated primarily by convenience. This method of organization recommends itself through the brevity of the proofs and the resulting compactness of the entire work, which make it attractive for both study and reference. The price we must pay is that more than a few times a theorem which is applied is not proved. This is certainly excusable in the case of the Corona Theorem, Carleson measures and interpolating sequences; it is less so in the case of the closedness of $H^\infty + C$ (used to prove Hartman's fundamental theorem in Chapter 1) and Douglas' Localization Theorem.

A missed opportunity is the omission of Rosenblum's Hilbert matrices [**22**]. Their spectra could easily have been obtained as an attractive corollary of the theorem on the s-numbers and the author's result on the essential spectrum for φ piecewise continuous.

The historical notes "do not supply full credits", according to the introduction. In fact, they occasionally err in the direction of generosity, so that the reader may at times lose sight of Power's considerable personal contribution to the subject.

A class of operators which has been found to arise in a variety of problems and which is worthy if further study in its own right is now available in convenient form for the student or the researcher in related fields.

REFERENCES

1. V. M. Adamjan, D. Z. Arov and M. G. Krein, *Infinite Hankel matrices and generalized Carathéodory-Fejér and Riesz problems*, Funkcional Anal. i Prilozen **2** (1968), 1–19 = Functional Anal. Appl. **2** (1968), 1–18.

2. ______, *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Shur-Takagi problem*, Mat. Sb. **86(128)** (1971), 34–75 = Math. USSR Sbornik **15** (1971), 31–73.

3. S. Axler, I. D. Berg, N. Jewell and A. Shields, *Approximation by compact operators and the space* $H + C$, Ann. of Math. (2) **109** (1979), 601–612.

4. S. Axler, S.-Y. A. Chang and D. Sarason, *Products of Toeplitz operators*, J. Integral Equations Operator Theory **1** (1978), 285–309.

5. C. Carathèodory and L. Fejér, *Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landauschen Satz*, Rend. Circ. Math. Palermo **32** (1911), 218–239.

6. D. N. Clark, *On the spectra of bounded, Hermitian Hankel matrices*, Amer. J. Math. **90** (1968), 627–656.

7. ______, *On interpolating sequences and the theory of Hankel and Toeplitz operators*, J. Functional Analysis **5** (1970), 247–258.

8. L. A. Coburn, *The C^*-algebra generated by an isometry*. II, Trans. Amer. Math. Soc. **137** (1969), 211–217.

9. R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in* L^p, Astérisque **77** (1980), 11–66.

10. I. C. Goldberg and N. Ja. Krupnik, *The algebra generated by Toeplitz matrices*, Funkcional Anal. i Prilozen **3** (1969), 45–56.

11. P. Hartman, *On completely continuous Hankel matrices*, Proc. Amer. Math. Soc. **9** (1958), 362–366.

12. P. Hartman and A. Wintner, *On the spectra of Toeplitz's matrices*, Amer. J. Math. **72** (1950), 359–366.

13. J. S. Howland, *Trace class Hankel operators*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 147–159.

14. M. G. Krein, *Integral equations on a half line with difference kernel*, Uspehi Mat. Nauk (N.S.) **13** (1958), No. 5 (83), 3–120.

15. W. Magnus, *On the spectrum of Hilbert's matrix*, Amer. J. Math. **72** (1950), 699–704.

16. Z. Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162.

17. V. V. Peller, *Hankel operators of the class γ_p and their applications*, Mat. Sb. (N.S.) **113 (155)** (1980), 538–581.

18. S. C. Power, *The essential spectrum of a Hankel operator with piecewise continuous symbol*, Michigan Math. J. **25** (1978), 117–121.

19. ______, *C*-algebra generated by Hankel operators and Toeplitz operators*, J. Functional Analysis **31** (1979), 52–68.

20. ______, *Hankel operators with PQC symbol and singular integral operators*, Proc. London Math. Soc. **40** (1980), 45–65.

21. C. R. Putnam, *Commutators and absolutely continuous operators*, Trans. Amer. Math. Soc. **87** (1958), 513–525.

22. M. Rosenblum, *On the Hilbert matrix*. I, II, Proc. Amer. Math. Soc. **9** (1958), 137–140; **9** (1958), 581–585.

23. ______, *The absolute continuity of Toeplitz's matrices*, Pacific J. Math. **10** (1960), 987–996.

24. T. Takagi, *On an algebraic problem related to an analytic theorem of Carathéodory and Fejér and on an allied theorem of Landau*, Japanese J. Math. **1** (1924), 83–93.

25. A. L. Volberg, *Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason*, J. Operator Theory **7** (1982), 209–218.

DOUGLAS N. CLARK

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

0273-0979/83 $1.00 + $.25 per page

Basic concepts of enriched category theory, by G. M. Kelly, London Mathematical Society Lecture Note Series, No. 64, Cambridge Univ. Press, New York, 1982, 245 pp., $24.95. ISBN 0-5212-8702-2

What has been going on in category theory for the last 15 years? Originally category theory appeared to be an outgrowth of homological algebra which itself developed as an aspect of algebraic topology. The historical accident of its birth has little to do, however, with the current perception of category theory as an alternative to set theory in the foundations and formulations of mathematics. What originally distinguished category theory from homological algebra was its retreat from groups to sets; i.e., its elimination of the requirement that maps between the same two objects could always be added. This ruled out exact sequences and hence derived functors so that for a while category theory not only bore little relation to homological algebra, but also had little to talk about. Fortunately adjoint functors were discovered (remembered? recognized?) and they became the main topic of study in the 1960's, which mostly centered around the notions of triples (monads), algebras for a triple, algebraic categories and equational categories. The names of Barr, Beck, Lawvere, and Linton are associated with this development, which had its culmination in the books by Gabriel-Ulmer [**6**] and Manes [**21**]. (See these for references to the original papers.) However, during the 60's there were other developments that could be characterized as "moving away from the safe shores of set theory". (Thanks to F. E. J. Linton for this image.) This movement took place in three interrelated ways.

1. Closed categories. Being able to add maps between the same two objects is clearly an important property of a category. This can be expressed in a more abstract form by requiring that for any two objects A and B in the category $\mathbf{C}$, the set $\mathbf{C}(A, B)$ of maps from A to B in $\mathbf{C}$ should carry the structure of an