I enjoyed reading this book. While presenting a lot of mathematics, carefully and in considerable detail, Davis has managed to be conversational whenever possible, and to tell us where he's going and why. A class of advanced undergraduates with good linear algebra preparation should be able to read this book, to work the exercises, and to discuss their work together. Even though circulant matrices are themselves not of primary importance, their study, via this excellent book, could be a valuable part of discovering mathematics as a discipline.

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Essays in commutative harmonic analysis, by Colin C. Graham and O. Carruth McGehee, Grundlehren der Mathematischen Wissenschaften, Band 238, Springer-Verlag, New York, 1979, xxi + 464 pp., $42.00.

Harmonic Analysis has passeed through a series of distinct epochs. The subject began in 1753 when Daniel Bernoulli gave a "general solution" to the problem of the vibrating string, previously investigated by d'Alembert and Euler, in terms of trigonometric series. This first epoch continues up to Fourier's book (1822). It is not completely clear what was going on in the minds of the harmonic analysts of this period, but the dominant theme seems to me to be the spectral theory of the operator d^2/dx^2 on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

In 1829 Modern Analysis emerges fully armed from the head of Dirichlet [3]. His proof that the Fourier series of a monotone function converges pointwise everywhere meets the modern standards of rigor. All needed concepts are defined, a marked contrast with previous habits, and the demonstration is expeditious. The operator d^2/dx^2 does not enter. The next epoch-making work is Riemann's *Habilitationschrift* [6]. Once more d^2/dx^2 plays the central role, albeit in a generalized form,

$$D^2_{\mathrm{sym}} f(x) = \lim_{n\to\infty} h^{-2}[f(x+h) + f(x-h) - 2f(x)].$$

The big gap in Riemann's work is the Uniqueness Theorem, proved by Cantor [1] in 1870 after Schwarz has shown that the solution to $D^2_{\mathrm{sym}} f = 0$ are harmonic functions in the ordinary sense. After this d^2/dx^2 goes its own way. With the Lebesgue integral and the pioneering work of Herman Weyl there has been a main branch of harmonic analysis going on for seventy years or so where the principal theme has been the study of various group-invariant generalizations of d^2/dx^2. Almost all of what is called "harmonic analysis on Lie groups" is in this vein. Moreover, this is the branch of the subject where exciting new ideas are still pouring forth. If one looks at what the leaders in the field have been doing recently, in the majority of cases one finds that harmonic analysis has more to do with harmonic functions than with trigonometric series.

In the book under review d^2/dx^2 does not occur. Riemann's motivation for his work on trigonometric series was potential applications to the study of distributions of primes. One of the striking tours de force in the history of mathematics was Norbert Wiener's deduction of the Prime Number Theorem from a study of the convolution algebra $L_1(\mathbf{R})$. Wiener's 1933 textbook [**8**] treats $L_1(G)$, for $G = \mathbf{R}$, $\mathbf{Z}$, and the Bohr compactification of $\mathbf{R}$, in a spirit which anticipates the theory of commutative Banach algebras developed by Gelfand and others in the 1930's. In the period 1940–1960, the Banach algebra methods of abstract harmonic analysis were in vogue, and one learned to work on general locally compact abelian groups G with ease. Walter Rudin's text [**7**] appeared in 1962.

Graham and McGehee have written a sequel to Rudin's book at a more advanced level. After two introductory chapters Rudin begins with idempotent measures which, together with related matters, forms the subject of Chapter I of Graham and McGehee. The newer book has included, of course, results of more recent research. Ironically, the most important item missing is the proof of Littlewood's conjecture by McGehee, Pigno, and Smith [**4**], which came too late for inclusion in the book. The Littlewood conjecture is a lower estimate, $L_N \geqslant C \log N$, for

$$L_N = \inf \int_0^{2\pi} | c_1 e^{in_1\theta} + \cdots + c_n e^{in_N\theta} | \, d\theta/2\pi$$

where the infimum is taken over all N-tuples of integers $n_1 < n_2 < \cdots < n_N$, and complex constants $c_1, \ldots, c_N$ with $|c_k| \geqslant 1$. The case $n_k = k$, $c_k = 1$ shows that $L_N \leqslant A + B \log N$. (Littlewood posed the question for $c_k = 1$, but this is an unimportant restriction.) The fact that $L_N \to \infty$ as $N \to \infty$ leads to a characterization of the idempotent elements of $M(G)$, the convolution measure algebra. Nevertheless, the fundamental issue is a matter of estimating L_1-norms of trigonometric polynomials in terms of the coefficients. Paul Cohen [**2**] proved $L_N \geqslant C(\log N)^{1/8}$, and after twenty years of improvements of the estimates the conjecture was finally established. The proof given by McGehee, Pigno, and Smith is clever but easy, short, and elementary; it requires no knowledge of analysis nor any combinatorial tricks.

The word *Essays* at the beginning of the title means that the work is not a systematic treatise. Nevertheless, there are two main themes, and these are closely connected. One theme is Thin Sets. This goes back to Cantor's work on uniqueness mentioned earlier. Let $(c_n\colon n \in \mathbf{Z})$ be a sequence of complex numbers such that

$$\lim_{|n| \to \infty} c_n = 0 \quad \text{and} \quad \lim_{N \to \infty} \sum_{-N}^{N} c_n e^{inx} = 0 \quad \text{for } x \notin E$$

where E is a closed set (convergence could be replaced by suitable summability methods). Cantor's original theorem was $c_n \equiv 0$ if $E = \varnothing$. He soon saw that the same result was true if E was finite, or more generally, if $E' = \varnothing$ where E' is the derived set of E. The next step was to replace E' by $E^{(n)}$, the nth derived. Uniqueness still holds of $E^{(n)} = \varnothing$ where, as Cantor continued, n could be transfinite. Cantor abandoned Harmonic Analysis to found Set Theory; unfortunately, the subject of Sets of Uniqueness got to be overworked by others. As in the rest of Thin Sets, one finds mainly pathology: a countable closed set is a set of uniqueness; for a perfect set of measure zero it is nearly impossible to prove uniqueness without imposing arithmetical conditions on the set. A Helson set is a closed set $E \subset \mathbf{T}$ with the property that for every continuous function f on E there exists a sequence (a_n) with $\Sigma_{-\infty}^{\infty} | a_n | < \infty$ such that

$$f(x) = \sum a_n e^{inx} \quad \text{for } x \in E.$$

Such sets are very thin, and they are nearly (see Chapter 4) sets of uniqueness. T. Körner's example of a Helson set of multiplicity was, in my opinion, a fatal illness for Thin Sets.

The other main theme is $M(G)$. The Wiener-Pitt phenomenon, published in 1938, showed that the convolution algebra of bounded measures on G is not very nice as a Banach algebra when G is not discrete. Chapters 5 through 8 of Graham and McGehee give some constructive results, in the framework of J. L. Taylor's theory of commutative convolution measure algebras, and numerous examples of the pathology of $M(G)$.

Chapter 9 is an essay on functions which operate on Fourier transforms, Fourier-Stieltjes transforms, and positive-definite functions. Chapter 11 on Varopoulos' theory of Tensor Algebras and Chapter 12 on the algebra $\tilde{A}(E)$, the algebra of uniform limits of norm-bounded sequences of restrictions to E of Fourier transforms of L_1-functions, bring us back to thin sets again. Chapter 10 is an independent essay on L_p-multipliers. The spirit here is the comparison of the algebra of continuous L_p-multipliers with the algebra of Fourier-Stieltjes transforms. This is the only place in the book where there is any contact with the epoch, beginning about 1930 and now ending, of L_p, H_p, BMO-harmonic analysis on abelian groups.

The authors write very well. The proofs are usually elegant. Indeed, where the original proof of a theorem was very hard to follow, Graham and McGehee have either given simpler demonstrations or, where the route was still arduous, they have lucidly explained the outline of the proof and the key ideas before giving the details in a series of digestible lemmas. The book is a pleasure to read.

The work should become a standard reference for $M(G)$ and Thin Sets. It has an extensive bibliography and interesting comments on recent historical developments and guides for further reading. It is not a book to stand by itself. The reader ignorant of Kronecker sets will have difficulty in appreciating many of the remarks about them; Dirichlet sets occur without any definition being given. In my opinion, the book will not stimulate very much valuable research. A comparison with Kahane's monograph **[5]** of about nine years earlier is in order. The earlier work was written when positive results were being produced and Körner's counterexample will still in the future. Kahane sticks to the circle group **T** but he displays a wide variety of interesting techniques, including probabilistic methods (except for trivialities, these are avoided by Graham and McGehee). Kahane certainly inspired a lot of worthwhile work, but a decade later the subject has lost much of its interest. Not every answer deserves a new question. The past ten years have produced a spate of counterexamples in the type of harmonic analysis considered here. No doubt more will be forthcoming, and one can expect that the resolution of some of the unsolved problems listed by Graham and McGehee will show a lot of cleverness and ingenuity. But it is time to do something else. The general questions which can be posed in terms of locally compact abelian groups really come down to **T**, **Z** and **R**. There are still some things to do in $\mathbf{R}^n$. I would rather see some positive results related to nice subsets of $\mathbf{R}^n$ than more counterexamples for perfect nowhere dense subsets of $\mathbf{R}^1$. Also, commutative methods applied to noncommutative Lie groups have yielded some interesting results, but the authors say little about this.

In summary, Graham and McGehee have written an interesting monograph, and written it well, but it is an epitaph for an epoch in Harmonic Analysis, 1940–1980. Rest in peace.

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Operator algebras and quantum statistical mechanics, Volumes I and II, by Ola Bratelli and Derek W. Robinson, Springer-Verlag, New York-Heidelberg-Berlin; Volume I, *C^* and W^*-algebras, symmetry groups, decomposition of states*, 1979, xii + 500 pp., \$36.00; Volume II, *Equilibrium states, models in quantum statistical mechanics*, 1981, xi + 505 pp., \$46.00.

The theory of operator algebras was initiated by von Neumann in 1927, and Murray and von Neumann in 1936. One of the principal motivations of Murray and von Neumann for the theory was an application to a quantum