BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 2, September 1982

FACTORIZATION AND EXTRAPOLATION OF WEIGHTS

BY JOSÉ LUIS RUBIO DE FRANCIA

1. Introduction. The functions $w(x) \geqslant 0$ for which the Hardy-Littlewood maximal operator M is bounded in $L^p(w) = L^p(\mathbf{R}^n, w(x)dx)$, $1 < p < \infty$, are characterized by Muckenhoupt's A_p condition (see [5]). The description of all A_p weights in terms of A_1 weights (where $w \in A_1$ means $Mw(x) \leqslant Cw(x)$ a.e.) is given by the factorization theorem of P. W. Jones [7]

$$(1) \qquad w \in A_p \text{ iff } w = w_0 w_1^{1-p} \text{ for some } w_0, w_1 \in A_1.$$

The long and difficult proof of (1) depends heavily on the structure of cubes in $\mathbf{R}^n$ and on the very special properties of A_p weights (in particular: "$w \in A_p$ implies $w^s \in A_p$ for some $s > 1$"). We shall present here a different approach to (1) which is shorter and can be applied to more general weight classes. The ideas involved prove also some extrapolation theorems for weighted norm inequalities.

2. Statement of results. Let $(M_i)_{i \in I}$ be a family of positive operators in some measure space (X, dx) such that the maximal operator

$$Mf(x) = \sup_i |M_i f(x)|$$

is bounded in $L^p(dx)$ for all $p > 1$. If $1 < p < \infty$, we say that $w \in W_p$ when

$$\int Mf(x)^p w(x)dx \leqslant C_p(x) \int |f(x)|^p w(x)dx \qquad (f \in L^p(w))$$

while $w \in W_1$ means $Mw(x) \leqslant Cw(x)$ a.e. We make the following basic assumption:

$$(2) \qquad w \in W_p \text{ iff } w^{-p'/p} \in W_{p'} \qquad (1 < p < \infty).$$

THEOREM 1. *If $w \in W_p$, $1 < p < \infty$, there exists $w_0, w_1 \in W_1$ such that $w(x) = w_0(x) w_1(x)^{1-p}$.*

In particular, the theorem of P. Jones holds for the weights associated to the strong maximal function, Bergman projections [4, 3], martingales [6, 10] and ergodic theory [2].

Received by the editors February 23, 1982.
1980 *Mathematics Subject Classification.* Primary 42B25.

THEOREM 2. *Let T be a sublinear operator bounded in $L^r(w)$ for all $w \in W_{r/\lambda}$, where λ, r are fixed and $1 \leqslant \lambda \leqslant r < \infty$. Then*

$$\left\|\left(\sum_j |Tf_j|^q\right)^{1/q}\right\|_{L^p(w)} \leqslant C_{p,q}\left\|\left(\sum_j |f_j|^q\right)^{1/q}\right\|_{L^p(w)}$$

for all $w \in W_{p/\lambda}$ and all p, q with $\lambda < p, q < \infty$.

There is an analogous result for weak type operators. Some antecedents of Theorem 2 have appeared in [1, 7 and 8].

3. **Sketch of proofs.** If $w \in W_p$, the positivity of M and (2) imply

$$\left\|\left(\sum_j |T_j f_j|\right)\right\|_{L^p(w)} \leqslant C\left\|\left(\sum_j |f_j|\right)\right\|_{L^p(w)} \qquad (T_j \in \mathcal{T}) \tag{3}$$

where $\mathcal{T}$ consists of all positive linear operators whose adjoints are dominated by M. There is a general principle relating weighted norm inequalities and vector-valued inequalities (see [9]) which, applied to (3), gives, for each $u \in L_+^{p'}(w)$, some $U(x) \geqslant 0$ with

$$\|U\|_{L^{p'}(w)} \leqslant \|u\|_{L^{p'}(w)}$$

and

$$\int |Tf| uw \leqslant C \int |f| Uw \qquad (T \in \mathcal{T}).$$

To prove Theorem 1 when $1 < p \leqslant 2$, one defines inductively a sequence $\{u_j\}$ by $u_{j+1} = U_j + M_s u_j$, where $s = p'/p$ and $M_s u = M(u^s)^{1/s}$ (which is a bounded operator in $L^{p'}(w)$). If $c > 0$ is small enough, then $v(x) = \Sigma_j c^{-j} u_j(x)$ is finite a.e. and

$$\int |Tf| vw \leqslant (\text{const}) \int |f| vw \qquad (T \in \mathcal{T})$$

which is equivalent to $vw \in W_1$. Since $M_s v(x) \leqslant c^{-1} v(x)$ (i.e. $v^s \in W_1$) we have the desired factorization. When $p > 2$, one applies (2).

The same methods show that, if $w \in W_p$ and $1 \leqslant r < p$, then for every $u \in L_+^{(p/r)'}(w)$ there exists $v \in L_+^{(p/r)'}(w)$ such that $u(x) \leqslant v(x)$ and $vw \in W_r$. With this and Hölder's inequality, the case $q = r < p$ of Theorem 2 follows easily, and (2) can be used once more to prove the rest.

REFERENCES

1. K. F. Andersen and R. T. John, Studia Math. 69 (1980), 19–31.
2. E. Atencia and A. de la Torre, Studia Math. (to appear).
3. D. Bekollé, Studia Math. (to appear).
4. D. Bokollé and A. Bonami, C. R. Acad. Sci. Paris Sér. A 286 (1978), 775–778.
5. R. R. Coifman and C. Fefferman, Studia Math. 51 (1974), 241–250.

6. M. Izumisawa and N. Kazamaki, Tôhoku Math. J. 29 (1977), 115–124.

7. P. W. Jones, Ann. of Math. (2) 11 (1980), 511–530.

8. J. L. Rubio de Francia, Bull. London Math. Soc. 12 (1980), 211–215.

9. ———, Proc. Conf. Harmonic Analysis, (Minneapolis, 1981), Lecture Notes in Math. vol. 908, Springer-Verlag, Berlin and New York, 1982, pp. 86–101.

10. N. T. Varapoulos, J. Functional Anal. 39 (1980), 85–121.

DIVISION DE MATEMATICAS, UNIVERSIDAD AUTONOMA, MADRID, SPAIN