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NEW EXAMPLES OF MINIMAL IMBEDDINGS OF S^{n-1} INTO $S^n(1)$—THE SPHERICAL BERNSTEIN PROBLEM FOR $n = 4, 5, 6$

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The classical Bernstein theorem proves that an entire minimal graph in $\mathbf{R}^3$ is necessarily a plane. Analytically speaking, an entire minimal graph in $\mathbf{R}^{n+1}$ is given by an entire solution, $u(x^1, \ldots, x^n) \in C^2(\mathbf{R}^n)$, of the following minimal equation

$$\sum_{i=1}^{n} D_i \frac{D_i u}{\sqrt{1 + |Du|^2}} = 0.$$

The Bernstein problem asks whether an *entire* solution of the above equation is necessarily a *linear* function. The above problem was proved to be affirmative in the cases $n = 3$ by De Giorgi [6], $n = 4$ by Almgren [1] and $n \leqslant 7$ by Simons [9]. In the remaining cases of $n \geqslant 8$, it was settled to be negative by Bombieri, De Giorgi and Guisti in 1969 [2]. The study of Bernstein problem is closely related to that of minimal cones, singularities of minimal hypersurfaces and closed minimal hypersurfaces of the diffeomorphic type of $\mathbf{R}^{n-1}$ in E^n and that of the diffeomorphic type of S^{n-1} in $S^n(1)$. They are clearly simple testing problems of fundamental theoretical importance. For example, the following so-called spherical Bernstein problem was proposed by S. S. Chern in 1969 [4] and again in his address to International Congress of Mathematicians at Nice, 1970 [5] as an outstanding problem in differential geometry.

SPHERICAL BERNSTEIN PROBLEM. Let the $(n-1)$-sphere be *imbedded* as a minimal hypersurface in $S^n(1)$. Is it (necessarily) an equator?

The beginning case of $n = 3$ was known even before the above problem was proposed, namely, a theorem of Almgren [1] and Calabi [3]. So far, no progress has been made in the positive direction. We announce here the construction of infinitely many distinct new examples of *minimal imbeddings* of S^{n-1} into $S^n(1)$ for the cases $n = 4$, 5 and 6. Our construction makes use of the framework of

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equivariant differential geometry which reduces the analytical problem of nonlinear, parametric nature to a more manageable global problem of ordinary differential equation. We state the main results as follows

THE CONSTRUCTION. Let $(G, M) = (O(2) \times O(2), S^4(1)), (O(3) \times O(3), S^6(1))$ or $(SO(3), S^5(1))$ respectively, where G is the orthogonal transformation group fixing the north and south poles. Then the orbit space M/G is geometrically a *spherical lune* which can be conveniently represented by polar coordinate (r, θ) as follows

$$M/G = \{(r, \theta); ds^2 = dr^2 + \sin^2 r d\theta^2\}$$

$$\text{where} \begin{cases} 0 \leqslant r \leqslant \pi, 0 \leqslant \theta \leqslant \pi/2 & \text{for the first and the second cases,} \\ 0 \leqslant r \leqslant \pi, 0 \leqslant \theta \leqslant \pi/3 & \text{for the third case.} \end{cases}$$

It is easy to see that the geometry of (G, M) is symmetric with respect to both the r-bisector, $r = \pi/2$, and the θ-bisector, $\theta = \pi/4$ (resp. $\theta = \pi/6$ for the third case). Geometrically, the preimage of the r-bisector is the G-invariant equator $S^{n-1}(1)$, the preimage of the center point C (the intersection of the two bisectors) is the unique *minimal* G-orbit and the preimage of the θ-bisector is the suspension of the above minimal G-orbit, which is a minimal hypersurface with singularities at the north and south poles.

Schematically, one may picture the orbital geometry of (G, M) by the following figure

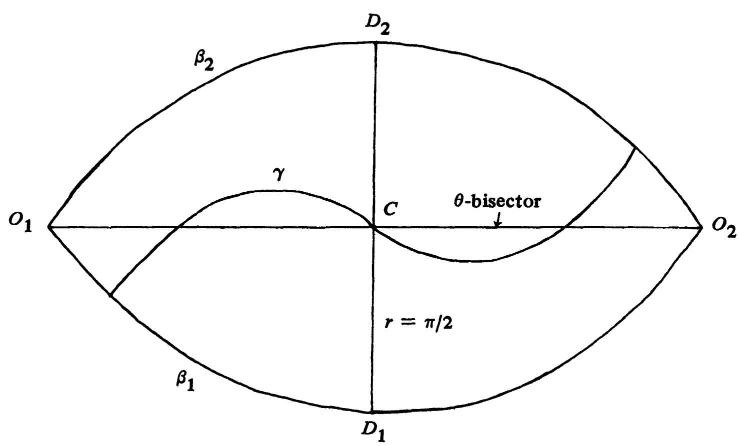

FIGURE 1

Following [7], one may reduce the analytical problem of finding G-invariant minimal hypersurfaces of certain type in $S^n(1)$, $n = 4, 5, 6$, by studying the geometry of solution curves of a specific nonlinear ODE with singularities. We state the main results of such construction as the following two theorems

THEOREM 1. *To each positive odd integer* $2i + 1$, *there exists a G-invariant, minimal imbedding of* S^{n-1} *into* $(G, S^n(1))$ *(for the above three cases) whose image-curve* $\gamma = S^{n-1}/G$ *is central symmetric with respect to the center point C and intersects with the* θ*-bisector at exactly* $2i + 1$ *points.*

Next let N be $S^2 \times S^1$, $S^3 \times S^2$ or the double of the mapping cylinder of $SO(3)/Z_2^2 \longrightarrow RP^2$ for the case $(O(2) \times O(2), S^4(1))$, $(O(3) \times O(3), S^6(1))$ or $(SO(3), S^5(1))$ respectively.

THEOREM 2. *To each positive even integer* $2i$, *there exists a G-invariant, minimal imbedding of N into* $(G, S^n(1))$ *whose image-curve* $\gamma = N/G$ *is reflectional symmetric with respect to the r-bisector and intersects with the* θ*-bisector at exactly* $2i$ *points.*

As $i \longrightarrow \infty$, the image curves of both Theorems 1 and 2 converge uniformly to the θ-bisector. Therefore, their corresponding minimal hypersurfaces converge to the suspension of minimal G-orbit as limit.

The proofs of the above two theorems and further discussion of the significance of such examples will be published elsewhere.

REFERENCES

1. F. J. Almgren, Jr., *Some interior regularity theorem for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) 85 (1966), 277–292.

2. E. Bombieri, E. De Giorgi and E. Guisti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969), 243–268.

3. E. Calabi, *Minimal immersions of surfaces in euclidean spheres*, J. Differential Geom. 1 (1967), 111–125.

4. S. S. Chern, *Brief survey of minimal submanifolds*, Tagungsbericht, Oberwolfach, 1969, pp. 43–60.

5. ———, *Differential geometry; its past and its future*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 41–53.

6. E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Sci. Norm Sup Pisa 19 (1965), 79–85.

7. W. Y. Hsiang and H. B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. 5 (1971), 1–38.

8. W. Y. Hsiang, Z. H. Teng and W. Yu, *New examples of constant mean curvature immersions of 3-sphere into euclidean 4-space*, Proc. Nat. Acad. Sci. U.S.A. (to appear).

9. J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62–105.

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