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THE WHITEHEAD CONJECTURE AND SPLITTING $B(\mathbf{Z}/2)^k$

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1. Introduction. In this note we present a circle of ideas with which the first author has proved G. Whitehead's conjecture concerning symmetric products of the sphere spectrum, i.e.

$$i_*\colon \pi_* SP^{2^k} S^0 \longrightarrow \pi_* SP^{2^{k+1}} S^0$$

is zero on the 2-components in positive dimensions [**Mi**, Conjecture 84]. Equivalently, the natural sequence of spectra

$$\cdots \longrightarrow L(3) \xrightarrow{\delta_2} L(2) \xrightarrow{\delta_1} L(1) \xrightarrow{\delta_0} L(0) \longrightarrow H\mathbf{Z},$$

localized at 2, is exact on homotopy groups. Here $H\mathbf{Z}$ is the integral Eilenberg-Mac Lane spectrum, $L(0) = S^0$, and $L(k) = \Sigma^{-k} SP^{2^k} S^0 / SP^{2^{k-1}} S^0$. Since $L(1) = \mathbf{R}P^\infty$ [**JTTW**], exactness at $L(0)$ is equivalent to the Kahn-Priddy theorem [**KP**].

In establishing this geometric resolution, it was found necessary to show that $L(k)$ is projective in an appropriate sense. Regarding suspension spectra as free objects, wedge summands of suspension spectra can be considered projective. The second and third authors have shown that $L(k)$ is projective by using the Steinberg idempotent [**S**] for $\mathbf{F}_2 GL_k(\mathbf{F}_2)$ to prove that $L(k)$ is a wedge summand in the suspension spectrum of $B(\mathbf{Z}/2)^k = \mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty$.

It appears likely that our results also hold true for odd primes and tentative results have been obtained in this direction. Throughout this paper all spaces and spectra are localized at 2 and all cohomology groups are taken with $\mathbf{Z}/2$ coefficients unless otherwise specified.

Details will appear elsewhere.

2. Symmetric products. If X is a space the *symmetric product* $SP^k X = X^k/\Sigma_k$ is the set of unordered k-tuples $\langle x_1, \ldots, x_k\rangle$, $x_i \in X$. For pointed X, $\langle x_1, \ldots, x_k\rangle \longrightarrow \langle x_1, \ldots, x_k, *\rangle$ defines an inclusion $SP^k X \xrightarrow{i} SP^{k+1} X$. The limit $SP^\infty X$ satisfies $\pi_* SP^\infty X = \widetilde{H}_*(X; \mathbf{Z})$ by the Dold-Thom theorem [**DT**]. There is also a natural pairing $SP^k X \wedge SP^l Y \xrightarrow{\wedge} SP^{kl}(X \wedge Y)$ defined by $\langle x_1, \ldots, x_k\rangle \wedge \langle y_1, \ldots, y_l\rangle \longrightarrow \langle x_1 \wedge y_1, \ldots, x_i \wedge y_j, \ldots, x_k \wedge y_l\rangle$. In particular $S^1 \wedge SP^k Y \xrightarrow{\wedge} SP^k(S^1 \wedge Y)$ and so the symmetric product construction passes to spectra. For the sphere spectrum, $SP^\infty S^0 = H\mathbf{Z}$. A mod 2

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version $SP_2^{2m}S^0$ can be defined as the cofiber of the diagonal map $SP^m S^0 \xrightarrow{\Delta} SP^{2m}S^0$. Then $SP_2^{\infty} S^0 = HZ/2$ and as a module over the Steenrod algebra, $H^*(SP_2^{2^k}S^0) = A/\langle Sq^{i_1} \cdots Sq^{i_l} : i_j \geqslant 2i_{j+1}, l > k \rangle$ [N].

3. Splitting $B(\mathbf{Z}/2)^k$ via the Steinberg module. Fix k and let $U = U_k$ be the unipotent group of upper triangular matrices in $G = GL_k(\mathbf{F}_2)$ and let $\Sigma = \Sigma_k$ be the Weyl group of permutation matrices. The *Steinberg idempotent* $e \in R = \mathbf{F}_2 G$ is defined by $e = \overline{U} \cdot \overline{\Sigma}$ where the bar indicates summation over all elements of the subgroup. The Steinberg module $St = e \cdot R$ is a projective, irreducible R-module of dimension $|U| = 2^{\binom{k}{2}}$. In fact, St belongs to a matrix ring block of R of dimension $|U|$ over $\mathbf{F}_2$. Thus there are $|U|$ orthogonal idempotnets $\{e_i\}$ associated with St.

Since $B(\mathbf{Z}/2)^k$ is 2-adically complete, the set of stable homotopy classes $[B(\mathbf{Z}/2)^k, B(\mathbf{Z}/2)^k]$ of maps is a module over the 2-adic group ring $\mathbf{Z}_2 G$. Lifting the Steinberg idempotents $\{e_i\}$ we obtain self maps $\{\widetilde{e}_i\}$ of $B(\mathbf{Z}/2)^k$ which provide a stable splitting

$$B(\mathbf{Z}/2)^k = \bigvee_i \operatorname{Tel}(\widetilde{e}_i) \vee \operatorname{Tel}(1 - \sum_i \widetilde{e}_i)$$

where Tel is the infinite mapping telescope.

Let $M(k)$ denote the quotient spectrum $\Sigma^{-k} SP_2^{2^k} S^0 / SP_2^{2^{k+1}} S^0$. Then $H^*(M(k))$, has a basis consisting of admissible Sq^I of length exactly k.

THEOREM (MITCHELL AND PRIDDY). *Stably, $B(\mathbf{Z}/2)^k$ contains $2^{\binom{k}{2}}$ summands each equivalent to $M(k)$. These summands correspond to the $2^{\binom{k}{2}}$ summands of the Steinberg module in $\mathbf{F}_2 GL_2(\mathbf{F}_2)$.*

A straightforward argument yields the

COROLLARY. $M(k) = L(k) \vee L(k-1)$.

Thus $B(\mathbf{Z}/2)^k$ also contains $L(k)$ as a stable summand.

A complete splitting

$$B(\mathbf{Z}/2)^2 = BA_4 \vee B\mathbf{Z}/2 \vee B\mathbf{Z}/2 \vee L(2) \vee L(2)$$

was obtained by the second author [M] and later extended by us [MP] to the dihedral group D_{2^k} of order 2^k:

$$BD_{2^k} = BPSL_2(\mathbf{F}_q) \vee B\mathbf{Z}/2 \vee B\mathbf{Z}/2 \vee L(2) \vee L(2)$$

where $\nu_2(q^2 - 1) = k + 1$. Partial results were also obtained by C. Witten [W].

SKETCH PROOF OF THE THEOREM. For $n \in \mathbf{Z}$ let P_n denote the stable projective space with bottom cell in dimension n [A]. Recall that $L(1) = P_1$. Similarly one shows $SP_2^2 S^0 = \Sigma P_{-1}$ and $M(1) = P_0 = B\mathbf{Z}/2_+$ where $+$ denotes addition of a disjoint base point. The pairing $\wedge$ of §2 passes to quotients and yields the commutative diagram

$$\begin{array}{ccc} \Sigma P_{-1} \wedge \cdots \wedge \Sigma P_{-1} & \xrightarrow{\wedge} & SP_2^{2^k} S^0 \\ \downarrow & & \downarrow \\ \Sigma^k(B(\mathbf{Z}/2)_+^k) = \Sigma P_0 \wedge \cdots \wedge \Sigma P_0 & \xrightarrow{\overline{\wedge}} & \Sigma^k M(k). \end{array}$$

Let $f\colon B(\mathbf{Z}/2)^k \longrightarrow M(k)$ denote the map induced by $\overline{\wedge}$. The theorem is proved by showing $e^*f^*\colon H^*(M(k)) \longrightarrow \operatorname{Im} e^*$ is an isomorphism.

4. The Whitehead conjecture. To state our main theorem we make the following definitions. A "split exact sequence of spaces" will denote a sequence of spaces of the following form, where all maps are the obvious composites of projections and inclusions.

$$\cdots \longrightarrow X_3 \times X_2 \longrightarrow X_2 \times X_1 \longrightarrow X_1 \times X_0 \longrightarrow X_0.$$

If E is a spectrum, let $\Omega^\infty E$ be the space obtained by taking the 0th space of an Ω-spectrum equivalent to E. Then Ω^∞ is the functor from spectra to spaces adjoint to the suspension functor. Let a sequence of spectra

$$\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$$

be called "exact" if application of the functor Ω^∞ to this sequence yields a split exact sequence of spaces.

MAIN THEOREM (KUHN). *The following sequence of spectra is exact.*

$$\cdots \longrightarrow \Sigma L(2) \longrightarrow \Sigma L(1) \longrightarrow \Sigma L(0) \longrightarrow \Sigma H\mathbf{Z}.$$

COROLLARY. *Whitehead's conjecture is true. In fact, if Y is any spectrum such that ΣY is a wedge summand of a suspension spectrum then the following sequence is exact:*

$$\cdots \longrightarrow [Y, L(2)] \longrightarrow [Y, L(1)] \longrightarrow [Y, L(0)] \longrightarrow [Y, H\mathbf{Z}] \longrightarrow 0.$$

The proof of the main theorem is rather indirect. We first construct spectra X_k with $H^*(X_k) \cong H^*(\Sigma L(k))$ and an exact sequence of spectra

$$\cdots \longrightarrow X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow \Sigma H\mathbf{Z}.$$

By viewing this sequence as being "acyclic" and interpreting the results of §3 as saying that the sequence of the main theorem is "projective," the two sequences can be compared, in analogy to the comparison theorem of homological algebra. The map between the sequences is then easily seen to be an equivalence, and the main theorem follows.

CONSTRUCTION OF THE SPECTRA X_k AND MAPS d_k. If X is a space, let $QX = \varinjlim \Omega^n \Sigma^n X$ and let $D_2 X = E\Sigma_{2+} \wedge_{\Sigma_2} X \wedge X$, the quadratic construction on X. Let $D_2^k S^1$ denote the k-fold iterated quadratic function on the circle

S^1. Inductively we construct stable maps f_k: $D_2^k S^1 \longrightarrow D_2^k S^1$ such that f_{k*} is idempotent in homology and $\operatorname{Im} f_{k*} \approx H_*(\Sigma L(k))$. We then let $X_k = \operatorname{Tel}(f_k)$, so that stably $D_2^k S^1$ contains X_k as a wedge summand, and construct $d_{k-1}: X_k \longrightarrow X_{k-1}$ as a composite of maps previously defined. To show that our sequence is exact, we use "transfer" maps defined to be the composites

$$\Omega^\infty X_k \longrightarrow QD_2^k S^1 \xrightarrow{j_2} QD_2^{k+1} S^1 \longrightarrow \Omega^\infty X_{k+1}$$

where j_2 is the James-Hopf map, arising from the stable splitting of $QD_2^k S^1$ **[K]**. Our main technical tool is the algorithm for computing j_{2*} in homology described in **[Ku]**.

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