BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 1, July 1982

RESEARCH ANNOUNCEMENTS

CLASSIFYING G SPHERES[1]

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Introduction. Let G be a finite group. The results announced here come from a study of the following general question: Classify all G actions on a sphere S, G homotopic to a given linear action.

This question has smooth, piecewise linear, and topological versions. Wall [W] solved the pl and topological problem, for free actions, when G is cyclic of odd order, and the dimension of the sphere is greater than 3. There are many partial results in the nonfree case. For example, if S is locally smooth, if dimension $S^G \geqslant 5$ and S satisfies the *mild gap condition* i.e. dimension S^{H_1} – dimension $S^{H_2} > 2$, for both nonempty and $H_1 \subsetneqq H_2$, then by G engulfing [I] S is topologically linear, and further if S is a pl G manifold, by G s-cobordism theorem [R] S is equivariantly pl determined by a generalized Whitehead torsion invariant.

In this note we announce some new results on this question.

Statements of results. In what follows G will always represent a cyclic group of odd order. We work in the locally linear i.e. locally smooth topological or pl category.

THEOREM A. *Locally linear* pl *or* top *G-vector bundles are oriented with respect to* $KO_G(\ ) \otimes Z[\frac{1}{2}]$.

From this, the methods of Schultz-Sullivan, cf. [S] and character theory one deduces easily the answer to the specific question which motivated our work.

THEOREM B. *Topologically conjugate representations of groups of odd order are linearly conjugate.*

Received by the editors December 1, 1981.

1980 *Mathematics Subject Classification.* Primary 57R65, 57Q65, 57R85; Secondary 20C15.

[1]Partially supported by NSF Grand MCS 8002730.

[2]This paper was scheduled to appear in the May 1982 issue of this *Bulletin*, simultaneously with the closely related paper "Orthogonal transformations for which topological equivalence implies linear equivalence" by W.-C. Hsiang and William Pardon (Bull. Amer. Math. Soc. (N.S.) 6 (1982), 459–461). However, the galleys were returned too late for the paper to appear in that issue owing to a misunderstanding concerning the deadline for the return of galley proof.

This theorem has also been proved independently using different methods by Hsiang and Pardon **[HP]**.

Theorem A has another interesting consequence, a pl G signature theorem. Consider a closed even dimensional pl G manifold M embedded in U, a complex G representation with pl normal bundle ϵ. Then by Theorem A we have an index map Φ_* which is the composition,

$$KO_G^*(M) \longrightarrow KO_G^*(\epsilon^+) \otimes Z[\tfrac{1}{2}] \longrightarrow KO_G^*(U^+) \otimes Z[\tfrac{1}{2}] \subset R(G) \otimes Z[\tfrac{1}{2}].$$

We also have the G signature of M denoted by $\mathrm{Sgn}_G(M)$. Further we have pl normal G block bundles $\gamma(M, M^g)$, for every $g \in G$. Since stably they are microbundles, by Theorem A they are $KO_G^* \otimes Z[\tfrac{1}{2}]$ oriented and the Euler classes $e(M, M^g)$ in $KO_G^*(M^g) \otimes Z[\tfrac{1}{2}]$ are defined.

THEOREM C. (i) $\mathrm{Sgn}_G(M) = \Phi_*(1)$.
(ii) $\mathrm{Sgn}_G(M)\,(g)(\Phi_* e(M, M^g))(g) = \mathrm{Sgn}(M^g)$, *and* $\Phi_*(e(M, M^g))(g) \neq 0$ *for all* $g \in G$ *if* $M^g \neq \varnothing$.

There is a weaker and more complicated version of Theorem C in the topological category. The motivation to look for a pl G signature theorem was inspired by conversations with Hsiang and Pardon for whom the proof of a pl version of the G signature theorem is a key step in their proof of Theorem B. In our approach it is a byproduct of the method and does not figure in our proof.

Finally there is a classification result. Recall from Wall **[W]** that if L is a homotopy Lens space, dimension $L \geqslant 5$, then L is determined up to pl equivalence by two invariants, a Reidemeister torsion invariant $\tau(L)$, and the ρ or η invariant; $\rho(L)$. If S is a pl G manifold, G homotopy equivalent to the unit sphere of a complex representation, then both invariants can be generalized. We then have the following:

THEOREM D. *If S satisfies the strong gap condition i.e.* $H_1 \subsetneqq H_2 \subset G$, $S^{H_1} \neq S^{H_2} \neq \varnothing$ *then dimension* $S^{H_1}/2 >$ *dimension* $S^{H_2} \geqslant 5$, *then S is determined up to G* pl *equivalence by* $\tau(S)$ *and* $\rho(S)$.

As in Wall we get more precise results. They are too technical to state explicitly here. If $S^G \neq \varnothing$ then $\rho(S)$ is trivial but $\tau(S)$ determines S as was already observed in **[R]**. Hence this theorem simultaneously generalizes both Rothenberg's and Wall's earlier results, at least under the hypothesis of the strong gap condition. We would like to replace the strong gap condition in Theorem D by the weak one but do not know how to prove it.

In the topological category ρ is still defined but τ is not a topological invariant. Theorem D, forgetting τ, is true up to a 2 torsion indeterminacy Λ, in some stable range.

Method of proof and auxillary results. Theorem A follows from an equivariant generalization of techniques of Connor-Floyd **[CF]** and a stable transversality theorem. Our main task is to prove such a theorem. In the pl category such a theorem holds without restriction, while in the topological category there is an obstruction, but it is a 2 torsion obstruction of a simple sort that we can control.

Our strategy is to first analyze the pl case using a pl variant of the equivariant surgery exact sequence developed in **[DR]**. Our main result here is an explicit calculation of the G homotopy type of the surgery space F/PL. The G action on F/PL is given by thinking of it as the fiber of $BPL(G) \longrightarrow BF(G)$, (see **[LR]**). We have

THEOREM E. *Let M satisfy the strong gap condition. Then*

$$[M/\partial M, F/\mathrm{PL}]_G \otimes Z[\tfrac{1}{2}] = \sum_{H \subset G}^{\oplus} KO_{G/H}(M^H, \partial M^H) \otimes Z[\tfrac{1}{2}].$$

This calculation is based on having a stable transversality theorem for the isotropy subgroups of M.

The reason that this result is linked to transversality is that G transversality can be reduced to questions of G homotopy via the theory of G submersions. This theory has been developed by Lashof along the lines of the G immersion theory as worked out in **[LR]**. In both pl and top, the problem of equivariantly deforming stable G maps to transversal ones can be reduced to the connectivity of the map $\mathrm{Aut}(V) \longrightarrow \mathrm{Aut}(V \oplus W)$, where V, W are representations of G and Aut means G automorphisms. We write $\mathrm{pl}(V)$ and $\mathrm{top}(V)$ for $\mathrm{Aut}(V)$ in the respective categories. We say V is stable if it satisfies the strong gap condition. In the topological case we need a more restrictive condition on V which we call super stability.

THEOREM F. *Suppose $V \subseteq T$ are stable representations and V and T have the same isotropy subgroups. Then* $\mathrm{pl}(V) \longrightarrow \mathrm{pl}(T)$ *is dimension $V^G - 1$ connected, while* $\mathrm{top}(V) \longrightarrow \mathrm{top}(T)$ *is dimension $V^G - 2$ connected. For $i =$ dimension $V^G - 1$ and T/V super stable and $V = W \oplus R$,*

$$\mathrm{image}(\pi_i(\mathrm{top}(T), \mathrm{top}(W))) \subset \pi_i(\mathrm{top}(T), \mathrm{top}(V))$$

is a finitely generated 2 group.

The proof of Theorem F in the pl case uses Theorem E for proper subgroups of G. The proof in the topological case uses the pl case and the methods of **[AH]** and **[LR]** to compare the two categories. In turn Theorem F for G implies Theorem E for G which implies Theorem D.

Let M, Y be locally linear G pl manifolds, $\epsilon \longrightarrow Y$ a locally linear G pl vector bundle. We say that (M, ϵ, Y) is stable if for all $H \subset G$, $x \in M^H$, $y \in Y^H$, $T_xM = V_{x,y} \oplus \epsilon_y$, where $V_{x,y}$, ϵ_y are stable representations of H and $\epsilon_y^H \neq 0$

implies $V^H_{x,y} \neq 0$. We have the analogous notion in the locally linear topological category. Theorem F now yields the following stable transversality theorem.

THEOREM G. *Let f: $M \longrightarrow \epsilon^+$ be a G map, where (M, ϵ, Y) is a stable (resp. super stable) triple of G manifolds.*

I. *In the* pl *case f is G homotopic to a G map transverse to the* 0 *section. The reasonable relative version is also valid.*

II. *In the topological case assume also $M^H \neq \emptyset$, simply connected. Using the connected sum of M with itself we can define nf for an arbitrary positive integer n, which also satisfies the same condition. Then $2^k f$ is G homotopic to a map transverse to the* 0 *section, for sufficiently large k. The reasonable relative version is also valid.*

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