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THE BRAVE NEW WORLD OF DETERMINANCY

The birth of descriptive set theory was marked by one of those curious events that occasionally act as a catalyst for an important discovery. An error found by a twenty year old student in a major work by a famous mathematician started a chain of theorems leading to the development of a new mathematical discipline.

For the background on the beginnings of the theory of analytic and projective sets let us go back to the early years of this century, to France, where Messrs. Baire, Borel and Lebesgue were laying foundations of modern function theory and integration [**5**, **2**, **16**, **6**]. A real-valued function of several real variables is a *Baire function* if it belongs to the smallest class of functions which contains all continuous functions and is closed under the taking of pointwise limits. A set in real n-space is a *Borel set* if it belongs to the smallest class of sets which contains all open sets and is closed under the taking of countable unions and intersections. Baire functions form a hierarchy, indexed by countable ordinal numbers: functions of Baire class 0 are the continuous functions, functions of Baire class 1 are the limits of sequences of continuous functions, and in general, functions of Baire class ξ are the limits of sequences of functions belonging to Baire classes smaller than ξ. Borel sets are similarly arranged in a hierarchy: sets of Σ-class 1 ($\mathbf{\Sigma}_1^0$) and Π class 1 ($\mathbf{\Pi}_1^0$) are, respectively, the open sets and the closed sets, and for each countable ordinal ξ, the class $\mathbf{\Sigma}_\xi^0$ (the class $\mathbf{\Pi}_\xi^0$) consists of all countable unions (countable intersections) of sets belonging to classes smaller than ξ. There is an intimate relationship between the hierarchies of Baire functions and of Borel sets; this relationship was extensively studied by Lebesgue in [**17**]. For instance, a function f: $\mathbf{R} \to \mathbf{R}$ is a Baire function if and only if for every interval (a, b), the set $\{x: a < f(x) < b\}$ is a Borel set.

Among others, Lebesgue showed that the hierarchy of Borel sets is a true hierarchy: for each countable ordinal ξ there is a Borel set of class ξ that does not belong to any smaller class; in fact there exists a $\mathbf{\Sigma}^0_\xi$ set which is not $\mathbf{\Pi}^0_\xi$. For small ξ, examples can be found in mathematical practice: for instance, the set of all rational numbers is $\mathbf{\Sigma}^0_2$ (also called F_σ in a different notation) but not $\mathbf{\Pi}^0_2$ (i.e. G_δ). For an arbitrary ξ, Lebesgue employed a method that generalizes Cantor's diagonal construction.

It was while reading Lebesgue's work **[17]** ten years later, that young Mikhail Suslin discovered an error on one of Lebesgue's proofs (cf. **[24]**). Suslin was a student of N. Luzin, who with a circle of young collaborators embarked on a systematic study of sets of real numbers. Another student of Luzin, P. Aleksandrov, had just proved **[1]** (and so did Hausdorff **[13]**), that every uncountable Borel set has the cardinality of the continuum. (For closed sets, this is the Cantor-Bendixson theorem **[3]**.)

The theorem with a false proof states that if a Baire function has an inverse, then the inverse is also a Baire function. The theorem is true and was subsequently proved by Luzin. Lebesgue however employed the following false lemma: If

$$S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$$

is a decreasing sequence of sets in the plane, then the projection to the x-axis of $\bigcap_{n=1}^{\infty} S_n$ is equal to the intersection of projections of the S_n. Suslin noted the fallacy of this argument, as well as of its corollary stated in Lebesgue's article: *The projection of a Borel plane set is a Borel set of reals.*

That statement is false: indeed, the projection of a G_δ set in the plane need not be a Borel set. Suslin went beyond proving Lebesgue wrong, he realized that he was dealing with an important property of sets of points. So he singled out the new class of sets, subsequently called *analytic* sets, namely the projections (or continuous images) of Borel sets. He laid the foundations of the theory of analytic sets in **[27]** (the only paper he wrote, and it was published after his death). Among others, he characterized analytic sets in terms of a set-theoretic operation (thereafter called *Suslin operation*) which appeared implicitly in Aleksandrov's proof mentioned above. From this characterization it was easily deduced, by Suslin himself and by others, that analytic sets are well behaved: every analytic set is Lebesgue measurable and has the Baire property (i.e. differs from an open set by a meager set), and every uncountable analytic set contains a perfect set and is thus of cardinality $2^{\aleph_0}$. The main result of **[27]**, the *Suslin theorem*, states that Borel sets are exactly those sets which are analytic and whose complement is also analytic.

Although the above discussion deals with sets in $\mathbf{R}^n$, the theory of analytic sets works equally well for any separable complete metric space. In fact, the prototype of such a space is the *Baire space* $\mathcal{N}$ which is just the set ω^ω of all infinite sequences of natural numbers with the product topology. For technical reasons, the Baire space is more in favor with descriptive set theorists than the real line, so I shall conform and confuse reals with functions $\alpha: \omega \to \omega$.

Luzin, Sierpiński and others developed Suslin's theory further, as follows: Let us call a set of reals *projective* if it is obtained from a Borel set by a finite number of projections and taking the complements. This defines the following hierarchy of sets (in modern notation):

$$\boldsymbol{\Sigma}^1_1 = \text{analytic sets}, \qquad \boldsymbol{\Pi}^1_1 = \text{complements of analytic sets},$$
$$\boldsymbol{\Sigma}^1_{n+1} = \text{projections of } \boldsymbol{\Pi}^1_n \text{ sets}, \quad \boldsymbol{\Pi}^1_{n+1} = \text{complements of } \boldsymbol{\Pi}^1_{n+1} \text{ sets}.$$

We also define

$$\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n.$$

Then Suslin's Theorem can be stated as

$$\boldsymbol{\Delta}^1_1 = \text{Borel}$$

and the projective sets form a hierarchy

$$\begin{array}{ccccccccccccc}
 & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & \Sigma^1_3 & & \\
 & \subset & & \supset & & \subset & & \supset & & \subset & & \supset & \\
\Delta^1_1 & & & & \Delta^1_2 & & & & \Delta^1_3 & & & & \cdots \\
 & \supset & & \subset & & \supset & & \subset & & \supset & & \subset & \\
 & & \Pi^1_1 & & & & \Pi^1_2 & & & & \Pi^1_3 & &
\end{array}$$

That this is a true hierarchy, with all the inclusions proper, can be shown by an extension of Lebesgue's method.

A prototypical example of a $\boldsymbol{\Pi}^1_1$ non-Borel set is the set of all well-orderings of ω (in the space of all binary relations on ω). The fact that being well ordered is a Π^1_1 property but not Σ^1_1 is central to modern descriptive set theory and numerous results are based on generalizations of the theory of $\boldsymbol{\Pi}^1_1$ sets. A related but more mathematical example is the following (due to Luzin) of a $\boldsymbol{\Sigma}^1_1$ set that is not Borel:

$$\{\alpha \in \mathcal{N}\colon \text{ there are infinitely many numbers among the } \alpha(0), \alpha(1), \alpha(2), \ldots \text{ that divide each other}\}.$$

The early study of projective sets was closely related to the problem of definability and "effectiveness" in mathematics, and particularly to the questions arising from the use of the axiom of choice. Ever since Zermelo's proof **[30]** that every set can be well ordered and his introduction of the axiom of choice there have been discussions among mathematicians whether indiscriminate use of the axiom of choice is legitimate and what constructions, particularly what constructions of sets of reals, can be done "effectively", without the use of the axiom of choice. Particularly vociferous were French analysts in the 20's and 30's and they objected mostly to *uncountable* choice, i.e. simultaneous choice from uncountably many sets. (One has to realize that even the elementary theory of measure and category requires *countable choice*: for instance, one has to make a countable number of choices even to prove the simple fact that the union of countably many countable sets of reals is a countable set.) An application of uncountable choice yields such "undesirable" results as a nonmeasurable set of reals, a set without the Baire property, or an uncountable set with no perfect subset, and there seems to be no "effective" construction of such examples.

Now clearly the concepts of definability and effectiveness are difficult to formulate in the language of mathematics alone and the problems of definability have eventually had to be handled by logicians. But it is also clear that the theory of projective sets, which is based on such simple concepts as open sets and continuous functions and using countable operations such as $\bigcup_{n=0}^{\infty}$ is a good approximation of what most mathematicians would consider "definable" or "effective". And this intuition of the early descriptive set theorists was remarkably confirmed by developments, some thirty years later, in a branch of mathematical logic called *recursion theory*.

* * *

In the meantime, descriptive set theory first flourished and then more or less came to a halt. Among the notable results were a detailed analysis of the structure of $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_2^1$ sets (showing for instance that every uncountable $\mathbf{\Pi}_1^1$ set must have cardinality either $\aleph_1$ or $2^{\aleph_0}$), theorems on reduction and separation (again dealing with $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_2^1$ sets) and the *uniformization theorem* of Kondô **[15]**: every $\mathbf{\Pi}_1^1$ set A in the plane contains a $\mathbf{\Pi}_1^1$ graph of a function which has the same projection as A. But there were no results on projective sets beyond level 2 and even simple questions about $\mathbf{\Sigma}_2^1$ remained unanswered. For instance: is every $\mathbf{\Sigma}_2^1$ set measurable?

There is a good reason why classical set theory cannot settle problems on projective sets beyond level 2. In 1938, Gödel gave his famous proof of consistency of the continuum hypothesis **[11]**. He constructed a model of set theory, the universe L of *constructible sets*, in which the continuum hypothesis holds, and consequently, it cannot be refuted from the axioms of set theory alone. (Years later, Cohen proved independence of the continuum hypothesis [7] by constructing a model in which the continuum hypothesis fails.) Gödel's model L has other remarkable properties, one of them being that there exists in L a well ordering of the set of all reals (in L)[1] which has order type $\aleph_1$ and is $\mathbf{\Delta}_2^1$. As a result of this property, the following theorems are true in L, and consequently cannot be refuted from the axioms of set theory:

(a) *There exists a* $\mathbf{\Delta}_2^1$ *set of reals which is not Lebesgue measurable and does not have the property of Baire.*

(b) *There exists an uncountable* $\mathbf{\Pi}_1^1$ *set of reals that has no perfect subset.*

In contrast, Cohen's proof provided a method which makes it possible to prove that it is consistent that every projective set is Lebesgue measurable and has the Baire property, and if uncountable, has a perfect subset (Solovay **[26]**).

These examples demonstrate the limitations of classical descriptive set

[1]A nonlogician might not be aware that unlike natural numbers, real numbers are not God-given (to paraphrase Kronecker), and it is not the case (or rather it cannot be proved) that every real is in the model L.

theory in the investigation of projective sets. Indeed, the extensive use of Cohen's method in the 60's and 70's established that many a natural question on projective sets is independent of the axioms of set theory. (A different kind of limitation is the availability of new methods: recent results of Martin [21] on the height of Σ_2^1 well-founded relations and of Silver [25] on Π_1^1 equivalence relations are in the spirit of classical descriptive set theory but their proofs use techniques not available twenty years ago.)

* * *

Those who worked in descriptive set theory were well aware of the connection between the theory of projective sets and logic. After all, the operations used in the construction of projective sets correspond to standard logical operations. To illuminate this connection, let me analyse the example given above, the set of all well orderings of the set ω of all natural numbers:

$$W = \{E \subseteq \omega \times \omega: E \text{ is a well ordering of } \omega\}.$$

A relation E belongs to W if

(i) E is a linear ordering of ω, and

(ii) there exists no infinite sequence $\alpha(0), \alpha(1), \ldots, \alpha(n), \ldots$ such that $\alpha(n+1) E \alpha(n)$ for all $n \in \omega$.

While the set of all linear orderings is a closed set in the space $\mathfrak{X}$ of relations, property (ii) states

$$\neg \exists \alpha \, \forall n (\alpha(n+1) E \alpha(n)).$$

Now the negation $\neg$ corresponds to the taking of complements, the existential quantification $\exists \alpha$ corresponds to projection (from $\mathfrak{X} \times \mathfrak{N}$ to $\mathfrak{X}$) and universal quantification $\forall n$ amounts to the taking of countable intersection $\bigcap_{n=0}^{\infty}$. It follows that W is the complement of the projection of a closed set in $\mathfrak{X} \times \mathfrak{N}$, and therefore a Π_1^1 subset of $\mathfrak{X}$.

This connection between descriptive set-theoretic and logical operations is just an aspect of the close relationship between the theory of projective sets and recursion theory. It is ironic that this close relationship remained unnoticed by the recursion theorists until after a considerable amount of work duplicating or parallel to the earlier work in descriptive set theory.

Like most other branches of modern logic, recursion theory originated in the work of Kurt Gödel [10]. In its simplest form, the theory of *recursive functions* attempts to capture the concept of "effectively computable" functions from ω to ω. A set of natural numbers is *recursive* if its characteristic function is a recursive function. In [14], Kleene extended recursion theory to functions ("functionals") whose arguments are functions $f: \omega \to \omega$. In particular, he introduced the notions of recursive sets in ω^ω and (partial) recursive functions from ω^ω to ω and from ω^ω to ω^ω. Most of the work of Kleene and his followers deals with a hierarchy of sets that resembles the hierarchy of Borel and projective sets. And a closer look reveals that indeed, Kleene's notions are refinements of the classical ones.

The notions of Kleene's recursion theory can be relativized to allow for parameters $\alpha \in \omega^\omega$; thus for each α one obtains sets and functions recursive in α. And it so happens that a function $f: \mathfrak{N} \to \mathfrak{N}$ is continuous if and only if it is recursive in some $\alpha \in \mathfrak{N}$. The analogy extends to projective sets. Let Σ_1^1 (lightface!) denote the class of sets of the form $\exists\alpha\,\forall nR$ where R is recursive. Then a set is $\mathbf{\Sigma}_1^1$ if and only if it is Σ_1^1 in some $\alpha \in \mathfrak{N}$. The classes Σ_n^1 and Π_n^1 are defined accordingly. (Incidentally, one of Kleene's major results was the theorem stating that Δ_1^1 = hyperarithmetical, which is the "effective form" of Suslin's theorem.)

Once the analogy between classical descriptive set theory and recursion theory was established, descriptive set theory became the domain of logicians. Not only does the logical symbolism simplify notation and clarify many complicated constructions, but the analogy makes available to the descriptive set theory techniques from recursion theory. An example is Kleene's recursion theorem which does not have an analog in the classical theory.

* * *

As a result of the work following Cohen's discoveries, it is more or less clear that not much more can be decided about projective sets in set theory alone. The major questions are independent of the axioms of set theory, and similarly as in the case of non-Euclidean geometries, one might consider various additional axioms that lead to different set theories, with different consequences for the theory of projective sets. A somewhat more promising approach is to study consequences for descriptive set theory of *large cardinals* axioms. These axioms postulate the existence of certain large cardinal numbers and are justified on philosophical grounds, as they increase rather than restrict the extent of the set-theoretic universe. Now large cardinals have a definite effect on the theory of projective sets. For example, if measurable cardinals exist, then every $\mathbf{\Sigma}_2^1$ set of reals is Lebesgue measurable and has the property of Baire **[26]**. But by and large, descriptive set theory would today be more or less at a standstill, if it was not for its extraordinary resurgence based on the study of consequences of a somewhat peculiar postulate, the *axiom of determinacy*.

The axiom of determinacy leads to an extremely fruitful theory, with remarkable consequences for the theory of projective sets. It is by no means a universally accepted axiom. Its unusual nature and the unique techniques associated with it make its theory strikingly different from the rest of set theory. The brave new world of determinacy is unlike our good old Cantor's universe; it's a strange world of infinite games, full of surprises for a conventional set theorist.

* * *

The first example of an infinite game was given by Mazur in 1935.[2] The Polish mathematicians in the city of Lwów (later to be annexed by the Soviet Union) were meeting regularly in the Scottish Coffee House and kept there a notebook in which they entered a number of interesting mathematical problems. This by now famous "Scottish Book" was eventually translated and made public by Ulam **[28]**. Under the title *Definition of a certain game* the

[2] In his book *Adventures of a mathematician* **[29]**, Ulam states: "In a conversation in the coffee house, Mazur proposed the first examples of infinite mathematical games".

book lists the following problem (#43) of Mazur: Let E be a set of real numbers. Consider a game between two players I and II where I selects an interval $[a_1, b_1]$, then II selects an interval $[a_2, b_2] \subseteq [a_1, b_1]$, then I selects an interval $[a_3, b_3] \subseteq [a_2, b_2]$ and so on. I wins if the intersection $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains a point in the set E; otherwise he loses. Is it true that there exists a method of winning for the player I for those and only those sets E that are comeager in some interval; similarly, does a method of win exist for II if and only if E is meager? Mazur's problem (which was solved by Banach–and presumably won for him one bottle of wine offered by Mazur) is a special case of a more general problem:

Let X be a fixed nonempty set. With each set A of infinite sequences from X, we associate a two-person game $G = G_X(A)$ as follows. Players I and II alternatively choose members of X ad infinitum: First, I chooses $a_0 \in X$, then II chooses $a_1 \in X$, then I chooses $a_2 \in X$, then II chooses $a_3 \in X$, and so on. Player I wins if the sequence $\{a_n\}_{n=0}^{\infty}$ is in A, otherwise II wins. A strategy for player I is any function σ whose agruments are all finite sequences from X of even length (including the empty sequence) with values in X. A strategy σ for I is a *winning strategy* if for any sequence $\{a_1, a_3, a_5, \ldots\}$ (of moves of player II) in X, the sequence

$$\begin{aligned} &a_0 = \sigma(\varnothing) \\ &a_1 \\ &a_2 = \sigma(a_0, a_1) \\ &a_3 \\ &a_4 = \sigma(a_0, a_1, a_2, a_3) \\ &\quad\ldots \end{aligned}$$

belongs to A (is a win for player I). A strategy (a winning strategy) for player II is defined similarly. The game $G_X(A)$ is *determined* if at least one player has a winning strategy (clearly, both players cannot have a winning strategy). The general problem is: what games $G_X(A)$ are determined? [It is well known that similarly defined *finite* games, i.e. games that end after a finite number of moves, are determined.]

Disregarding the general case, let us turn to the special case when either $X = \{0, 1\}$ or $X = \omega =$ the set of all natural numbers. Since these two cases are known to be practically equivalent, let us simply let $X = \omega$, and let us call a set $A \subseteq \omega^\omega$ *determined* if the game $G(A)$ is determined.

Axiom of determinacy (AD). *Every set $A \subseteq \omega^\omega$ of infinite sequences of natural numbers is determined.*

Granting the axiom of determinacy, one can formulate a multitude of infinite games for which one can then prove that they are determined. One example is the aforementioned Banach-Mazur game. Note that if that game is determined then every set of real numbers has the property of Baire. Using a different game, one can similarly deduce that every set of reals is Lebesgue measurable **[23]**, or that every uncountable set of reals has a perfect subset. Note that each of these statements is false in set theory with the axiom of choice, and so AD contradicts the axiom of choice. On the other hand, the *countable* axiom of choice (at least for sets of reals) is a *consequence* of AD.

The fact that the assumption of AD eliminates some "unpleasant" consequences of the axiom of choice is certainly notable and perhaps with that in mind Mycielski and Steinhaus proposed in [22] to replace the axiom of choice by the axiom of determinacy in the development of mathematics. Although this proposal has not been accepted (even the majority of those working with AD do not consider it a "true" principle), subsequent results and methods associated with the axiom made AD an attractive subject of study.

One remarkable feature of the axiom of determinacy is its connection with the theory of large cardinals. The first indication of this was Solovay's result, in 1967, showing that under AD, $\aleph_1$ is a measurable cardinal. In ordinary set theory (with the axiom of choice), measurable cardinals are large ($\aleph_1$ is too small to be measurable), but it is the existence of a measurable cardinal (even if it is $\aleph_1$) that provides the connection with the theory of large cardinals. For one can then construct a model of ZFC, the Zermelo-Fraenkel axiomatic system with the axiom of choice, that has a measurable cardinal. This link between determinacy and large cardinals has been considerably extended and exploited. One consequence is that one cannot hope to establish the consistency of AD without assuming some (rather strong) large cardinals hypotheses.

There is even a stronger link between determinacy and projective sets. Already Solovay's result makes an essential use of a certain basic property of $\mathbf{\Pi}_1^1$ sets (the boundedness lemma). It was however Blackwell's paper [4] that firmly established the game-theoretic method in descriptive set theory. Blackwell gave a new proof of this theorem of Kuratowski: If A and B are $\mathbf{\Pi}_1^1$ sets of reals, then there are disjoint $\mathbf{\Pi}_1^1$ sets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A_1 \cup B_1 = A \cup B$ (the *reduction principle* for $\mathbf{\Pi}_1^1$ sets). In the proof, he used the known fact [9] that closed games are determined. As the method of proof is quite general, one can, for instance, use AD to extend the reduction principle to higher levels of the projective hierarchy.

Much of the current research on AD deals with extending the theory of $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_2^1$ sets to higher levels of the projective hierarchy and beyond. It has been established that the classes $\mathbf{\Pi}_n^1$ with odd n and the $\mathbf{\Sigma}_m^1$ with even m have very similar properties as the class $\mathbf{\Pi}_1^1$ (or $\mathbf{\Sigma}_2^1$) and the classes $\mathbf{\Pi}_n^1$ with even n and the $\mathbf{\Sigma}_m^1$ with odd m behave similarly as the class $\mathbf{\Pi}_2^1$. In fact, there is a transfinite hierarchy of *pointclasses* (classes of sets of reals) extending well beyond projective sets and many properties of $\mathbf{\Pi}_1^1$ sets (such as uniformization) can be proved for these pointclasses.

Compared to Cantor's universe, the world of determinacy (or at least its part comprising of sets of reals) is remarkably structured. The loss of the axiom of choice is amply compensated by the construction principles derived from existence of winning strategies in various infinite games. So for example many sets of reals admit a representation similar to Suslin's operation. Or, consider this: if A and B are sets of reals, we say that A is *reducible* to B if there is a continuous function f such that $A = f_{-1}(B)$. Now AD implies that for *any* two sets of reals A and B, either A is reducible to B or B is reducible to the complement of A.

What is fascinating on the world of determinacy is the interplay of set-theoretic, game-theoretic and recursion-theoretic methods. These methods include such diverse techniques as ultrapowers and partition properties on one hand and the recursion theorem and inductive definability on the other. But it is not an easy world to grasp. For instance: what intuition could possibly be behind the fact that $\aleph_1$ and $\aleph_2$ are measurable cardinals, but for every $n \geqslant 3$, $\aleph_n$ is a singular cardinal (of cofinality $\aleph_2$)? And $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ are again measurable.

As mentioned above, all closed sets $A \subseteq \mathfrak{N}$ are determined. The natural question arises whether AD holds for, say, all projective sets. Besides, determinacy of projective sets does not seem to contradict the axiom of choice and so the *axiom of projective determinacy* naturally offers itself as an axiom for descriptive set theory, more plausible (albeit less powerful) than AD. However, despite considerable progress in this area, the main question, whether projective determinacy is consistent, is still unresolved. The present state of knowledge is this: All Borel sets are determined (Martin **[19]**). And $\mathbf{\Pi}_1^1$- and $\mathbf{\Pi}_2^1$-determinacy are essentially large cardinals axioms; more precisely, $\mathbf{\Pi}_1^1$-determinacy is equivalent to a certain standard large cardinals axiom, while $\mathbf{\Pi}_2^1$-determinacy follows from another, stronger, standard large cardinals axiom (Martin **[18]**, Harrington **[12]**, Martin **[20]**). The consistency of $\mathbf{\Pi}_3^1$-determinacy is still an open problem and so until proven consistent, the axiom of projective determinacy has to be considered a speculative axiom.

* * *

It is this world of determinacy that Moschovakis explores in his book. Although determinacy itself is not investigated until Chapter 6, everything in the book indicates that the author's view of descriptive set theory is biased toward infinite games. From the early pages of the book, the development of the general theory, and even the selection of theorems of classical descriptive set theory is definitely tailored to be used later under the assumption of determinacy. Those descriptive set theorists who are less enthusiastic about determinacy may object that results not directly related to determinacy are either omitted or played down but the book does not claim completeness, and besides, its coverage of the basic material in descriptive set theory is adequate. [The author deliberately omitted those parts of descriptive set theory that use forcing (a phobia of forcing?) or admissible sets.] The book is extremely well organized and shows that the author put many years of work into writing the book.

* * *

The underlying assumption throughout the book is that infinite games on natural numbers are determined. For the author, this is not just a technical assumption. In his (platonist) view of the set-theoretical universe, assertions on infinite sets are either true or false. And this presents for him a dilemma: since he believes that the axiom of choice is true, the axiom of determinacy must be false. Moschovakis resolves this dilemma by replacing the axiom of determinacy by the assumption that AD holds in the model $L(\mathfrak{N})$ of all sets constructible from the reals. (Formally, this distinction is immaterial, as the consistency of the substitute assumption is equivalent to the consistency of AD.)

The way the author treats the axiom of choice is a logical extension of these assumptions. If the axiom of choice holds in the universe then the model $L(\mathcal{N})$ satisfies the countable axiom of choice and, in fact, a somewhat stronger *principle of dependent choices* (PD). And so while it is explicitly pointed out whenever the full AC is used, and even a list is made of these uses, the countable axiom of choice (and PD) is used throughout the text without any reference. This is somewhat disappointing, in particular in the development of the Borel hierarchy where only a passing allusion is made to the countable axiom of choice. After all, the problems related to definability are central to descriptive set theory, and the question of what form of the axiom of choice one should use was once hotly disputed. The main reason why the countable axiom of choice is (unlike the full AC) widely accepted by descriptive set theorists is that it is indispensable: it is for instance necessary for the proof of closure of $\mathbf{\Sigma}_2^0$ sets under countable unions. (In the model **[8]** of Feferman and Lévy, the set of all reals is the union of countably many countable sets.)

* * *

With very few exceptions, Moschovakis' writing is rigorous but readable. A notable departure from his otherwise meticulous style are the exercises in Chapter 8G. There the author assembled a collection of very interesting new results, but the less polished presentation and some unfortunate misprints make this fascinating material less accessible to the nonexpert reader. Otherwise, there are very few misprints (which one hopes will be corrected in the second printing).

I like the book; it is on my list of books that I would take with me, if not to a deserted island, then at least when going on a sabbatical. What I do not like about the book is its price. Almost $75 is a lot of money to pay for a book, even for a fine book like this one.

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Hopf algebras, by Eiichi Abe, Cambridge Univ. Press, 1980, translated by Hisae Kinoshita and Hiroko Tanaka, xii + 284 pp., $39.50.

The Hopf algebras under consideration are not the graded coalgebras/Hopf algebras of algebraic topology. Rather these are the Hopf algebras whose study was motivated by such examples as group algebras, universal enveloping algebras of Lie algebras and representative functions on Lie groups and Lie algebras [**5**, **6**, **8**]. Indeed the emphasis in Abe's book is on Hopf algebras which are either commutative or cocommutative. About twelve years ago another book by the same title appeared [**14**]. Since the first *Hopf algebras* was published, some open questions in Hopf algebra theory have been answered, and coalgebras/Hopf algebras have enjoyed wide application. This book presents the answer to one of these questions—uniqueness of Hopf algebra integrals. It presents two important areas of Hopf algebra applications—to algebraic groups and to field theory. A book like Abe's *Hopf*