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ON THE HOPF INDEX THEOREM AND THE HOPF INVARIANT[1]

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Let $f: N \longrightarrow M$ be a C^∞ map of oriented compact manifolds, and let L be an oriented closed submanifold of codimension $q \geqslant 1$ in M. If w is a closed form Poincaré dual to L, we show that $f^{-1}L$, with multiplicities counted, is Poincaré dual to f^*w in N and is even meaningful on a "secondary" level. This leads to generalized versions of the Hopf invariant, the Hopf index theorem and the Bezout theorem.

We assume that the connected components $\Gamma_1, \ldots, \Gamma_l$ of $f^{-1}L$ are submanifolds of codimension q in N. Let ord Γ_i be the intersection number of L and $f|B$, where B is a q-dimensional submanifold meeting Γ_i transversally at a single point. A proper choice of orientations makes ord $\Gamma_i \geqslant 0$.

THEOREM 1. *The cycle* $\Sigma(\text{ord } \Gamma_i)\Gamma_i$ *is Poincaré dual to* f^*w.[2]

This assertion improves a known theorem, which requires that f is transversal to L and, consequently, ord $\Gamma_i = 1$.

THEOREM 2. *Let w' be an integral closed q'-form on M with $q + q' - 1 > \dim M$. If both f^*w and f^*w' are exact with $f^*w' = du$ on N and if σ is a closed p-form on N with $p + q + q' = \dim N$, then*

$$\int_N f^*w \wedge u \wedge \sigma = \sum (\text{ord } \Gamma_i) \int_{\Gamma_i} u \wedge \sigma.$$

COROLLARY. *Let $f: N \longrightarrow M$ be an arbitrary C^∞ map (without any condition on $f^{-1}L$). If r is the least positive integer making $rH_{q+q'-1}(N; Z)$ free abelian, then the cohomology class of $rf^*w \wedge u$ is integral.*

A sketched proof of Theorem 1 runs as follows. There exists a $(q - 1)$-form v on $M - L$ with $dv = w|M - L$. Let σ be a closed p-form on N such that

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[2] The author wishes to thank the referee for pointing out the general validity of Theorems 1 and 2. Our proof of these theorems is a modification of the original version, which, however, already suffices to cover Theorem 3 and other applications in this note.

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$p + q = \dim N$. Then, on $N - \Gamma$, $d(f^*v \wedge \sigma) = f^*w \wedge \sigma$. Let $T_i(\epsilon)$ be the ϵ-tubular neighborhood about Γ_i. Then

$$\int_N f^*w \wedge \sigma = -\sum \lim_{\epsilon \to 0} \int_{\partial T_i(\epsilon)} f^*v \wedge \sigma.$$

The theorem will follow from the formula

$$\lim_{\epsilon \to 0} \int_{\partial T_i(\epsilon)} f^*v \wedge \sigma = -(\text{ord } \Gamma_i) \int_{\Gamma_i} \sigma. \tag{1}$$

This formula is verified by using, in essence, the approach described below. The integration over $\partial T_i(\epsilon)$ is first performed over fibers and then over Γ_i. Since σ is defined everywhere on N, its contribution during the integration over fibers can be ignored as $\epsilon \longrightarrow 0$. The fiber of the ϵ-tubular neighborhood $T_i(\epsilon)$ is a q-dimensional submanifold $B(\epsilon)$ transversal to Γ_i. We show that ord $\Gamma_i = -\lim_{\epsilon \to 0} \int_{\partial B(\epsilon)} f^*v$ and thus (1).

In order to prove Theorem 2, we take note that $v \wedge w' = 0$ on $M - L$ and that $d(f^*v \wedge u) = f^*w \wedge u$ on $N - \Gamma$. The theorem follows from the Stokes theorem and the formula

$$\lim_{\epsilon \to 0} \int_{\partial T_i(\epsilon)} f^*v \wedge u \wedge \sigma = -(\text{ord } \Gamma_i) \int_{\Gamma_i} u \wedge \sigma.$$

Realizing that w' can be modified so that w' vanishes on a neighborhood of L, we can verify this formula in the same way as for (1).

EXAMPLE 1. Let M be the q-sphere bundle obtained by compactifying a C^∞ vector bundle of fiber dimension q over an oriented compact manifold of dimension $< q - 1$. Let L_0 and L be respectively the zero and the "infinity" sections of M. Let $f\colon N = S^{2q-1} \longrightarrow M$ be a C^∞ map such that both $f^{-1}L_0$ and $f^{-1}L$ are smooth and of codimension q in N. Let $f^*w = du$ and let $\Sigma(\text{ord } \Gamma_i)\Gamma_i = \partial Z$ in N. According to Theorem 2,

$$\begin{aligned}\int_{S^{2q-1}} f^*w \wedge u &= \int_Z f^*w \\ &= \text{the algebraic linking number of } f^{-1}L \text{ and } f^{-1}L_0.\end{aligned} \tag{2}$$

The Whitehead integral formula for the Hopf invariant [3] is thus valid in this generalized situation.

EXAMPLE 2. Let M be the fiber bundle obtained from a C^∞ complex (or quaternion) vector bundle by replacing each fiber with its complex (or quaternion) projective compactification. Let L_0 be the zero section and let L be the union of the hyperplanes at infinity of the fibers. Theorem 1 implies that, under reasonable conditions, the homology class of the poles of a C^∞ section of M does not depend on the choice of the section.

EXAMPLE 3. Let $M = G(k, C^n)$ be the Grassmannian of complex k-planes in C^n, $n > k$. Let $L = G(k, C^{n-1})$. Then the kth Chern form w of M is Poincaré dual to L. Set $w' = w^{n-k}$. It can be verified that, if N is the associated $(2k - 1)$-sphere bundle of the universal vector bundle of M and if $f: N \longrightarrow M$ is the bundle map, then $f^*w' = du$ and $\int_N f^*w \wedge u = 1$.

Let $E \longrightarrow N$ be a C^∞ complex k-plane bundle and let $s: N \longrightarrow E$ be a C^∞ section having Δ as its zero set whose connected components are $\Delta_1, \ldots, \Delta_l$. We assume that each Δ_i is smooth and of codimension $2k$ in N. The index of s along Δ_i is the integer $\text{index}_s\, \Delta_i$, which is the usual index of the restriction of s to a $2k$-dimensional submanifold meeting Δ_i transversally at a single point under a suitable choice of orientations. Theorem 1 leads to a generalized Hopf index theorem for complex vector bundles.

THEOREM 3. *The homology class of* $\Sigma\, (\text{index}_s\, \Delta_i)\Delta_i$ *is Poincaré dual to the kth Chern class of E.*

The main feature of this result is that the classical notion of index continues to provide multiplicities of the zero set Δ. Under certain generic conditions, there is a theorem of Griffiths [2] (see also [3, p. 413]) on Poincaré duals of all Chern classes of E. Theorem 3 extends a part of Griffiths' theorem to zero-sets with multiplicities.

Using Theorem 3, we point out a proof of a generalized version of the classical Bezout theorem. Let $z = (z_1, \ldots, z_n)$ be the coordinates of C^n, and let $g_1(z), \ldots, g_r(z)$ be polynomials of respective degrees $d_1, \ldots, d_r$, $r \leqslant n$. We assume that the common zero set Δ of $g_1, \ldots, g_r$ in CP^n is smooth and of pure codimension r. Let H be the hyperplane line bundle of CP^n. Then $(g_1, \ldots, g_r)$ is a section of $H^{d_1} \oplus \cdots \oplus H^{d_r}$, and so is $(z_1^{d_1}, \ldots, z_r^{d_r})$. Hence, with multiplicity counted, Δ must be homologous to $d_1 \ldots d_r(CP^{n-r})$.

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