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Operator colligations in Hilbert spaces, by Mikhail S. Livshits [Moshe Livsic] and Artem A. Yantsevich, Winston, Washington, D. C. (distributed by Wiley, New York), 1979, xii + 212 pp., $19.95.

The general philosophy behind the idea of *operator models* as a tool for studying a bounded linear operator on a Hilbert space is to associate with each operator some simpler objects which form a complete set of unitary invariants for the operator; next, one wants to give a recipe for constructing in a canonical way an operator on some concrete Hilbert space having the given invariants. Thus, for example, the first spectacular success for this approach was the spectral theorem for normal operators, due originally to von Neumann, which tells one how to build any normal operator (up to unitary equivalence) from its scalar spectral measure and its multiplicity function.

In the early 1950's, Livsic and his colleagues **[6]**, [7], **[18]** began developing such a structure theory for non-self-adjoint operators. The starting point was to embed the operator in an object called an *operator colligation* or more simply a *colligation* (sometimes translated in the older literature as *node* from the Russian original which literally means knot). The colligation by definition is a commutative diagram of linear maps and spaces which couples the original space H to an auxiliary space E which carries the metric (possibly indefinite) induced by the imaginary part of the original operator A (called the *principal operator* of the colligation). They described how invariant subspaces and extensions of the principal operator give rise to notions of factorization and prolongation for the colligation. They then introduced an analytic function of a complex variable, called the *characteristic function* of the colligation; the values of this function are operators on the auxiliary space E and are defined by a simple formula which involves transferring the resolvent of the principal operator to the auxiliary space via the maps of the colligation. This adds a complex-variables object to the formalism, which up to this point has been completely algebraic, and enables one to use techniques from complex variables to analyze colligations. Thus these early workers showed that the (algebraic) factorization of a colligation mentioned above corresponds to a factorization of the characteristic function (in the usual sense for functions). Standard representation theorems for Pick-class analytic functions and their operator-valued analogs then led to a spectral analysis for non-self-adjoint operators with trace-class imaginary part. Conversely, with the help of *triangular models* (both discrete and continuous versions) they were able to construct a completely non-self-adjoint operator as the principal operator of a colligation having a prescribed operator-valued function of the appropriate type as its characteristic function. Thus the characteristic function was the invariant from which one could recover the colligation; what developed here was really a model theory in the sense described above for the colligation rather than for the operator itself. All these developments are reviewed in the book of Livsic and Yantsevich (hereafter referred to as **[LY]**) with referrals to the original sources for proofs.

In the early 1960's the theme of characteristic functions appeared again but in a different way in the work of Sz.-Nagy and Foias (see **[22]** and the references there) and deBranges and Rovnyak **[4]**, **[5]**. Sz.-Nagy and Foias came upon their characteristic function for a contraction operator by studying the geometry of the unitary dilation space. They showed how to build a functional model which recovered the contraction operator and its unitary dilation space from the characteristic function, how *regular* factorizations of the characteristic function parametrized the invariant subspaces of the contraction operator, and how to describe the commutant via the lifting theorem. deBranges and Rovnyak developed a roughly equivalent model theory, but with the reproducing kernel function as the starting point. One could say that the Livsic and Sz.-Nagy-Foias theories gave a spectral theory for non-self-adjoint and contraction operators analogous to that achieved by von Neumann for normal operators. However, despite all this progress, none of these approaches (by those already mentioned as well as others, such as Helson [**11**]) was able to solve completely a basic structural problem: does a bounded linear operator on a separable Hilbert space have a nontrivial invariant subspace? Only recently Kriete [**16**] identified the completely non-self-adjoint part of Livsic's triangular model, and thus was able to read off that any dissipative operator with a trace-class imaginary part has a nontrivial invariant subspace, a result obtained in the meantime by other methods.

Thus, from the point of view of pure operator theory, the so-called operator "model theory" seemed to be an elaborate machine which was unable to solve any real problems. However, in the meantime the theory was branching out and making contact with engineering and physics. Livsic's two books [**19**], [**LY**] applied the colligation concept to engineering situations and nonstationary stochastic processes but remained relatively unknown in this country. Adamjan and Arov [**1**] discovered that the Lax-Phillips scattering theory [**17**] and the Sz-Nagy-Foias model had the same mathematical core, just with different points of view. Helton [**12**], [**13**] identified strong mathematical similarities between elements of systems theory (including scattering), electrical networks and operator models. Thus, as a few examples out of the many possible, the following are worth mentioning: such independently existing objects from physics and engineering as the scattering function and the frequency response function are really characteristic functions in disguised form, the whole Sz.-Nagy-Foias model theory can be reinterpreted as a treatise on infinite-dimensional discrete-time linear systems, and the Sz.-Nagy-Foias formula for the characteristic function of a contraction operator is really a special instance of cascade coupling of two electric circuits. All these developments have served to enrich the old model theory and suggest new directions and generalizations where the old single-operator point of view has pretty much been obliterated (see for example [**3**], [**14**]).

As suggested above, [**LY**] was one of the forerunners of this broadening of the ideas and techniques of "operator model theory" to diverse areas of applications. I shall limit the specific discussion of [**LY**] to three main themes: 1) open systems; 2) nonstationary stochastic processes; and 3) refinements of colligations and their characteristic functions.

1. *Open systems*. In classical physics, the state of a physical system at any point in time is described by a state vector in a linear space, the energy is given by a quadratic form on the state space, and energy is conserved as the system evolves in time. Completion of the state space in the energy norm gives rise to a Hilbert space, and the evolution of the system in time is described by a one parameter group of operators acting on the state space; since energy is conserved, this group is unitary, and then by Stone's theorem,

is generated by a self-adjoint operator. This explains the importance of self-adjoint operators in classical physics. The authors call such energy conserving systems *closed systems*, since they do not interact with the outside world. However, in practice many systems, especially those arising in engineering contexts, are not energy-conserving. The authors propose open systems (an idea also prominent in [**19**]) as a model for such systems. The state operator for an open system is now a non-self-adjoint operator; when one embeds it in a colligation, the auxiliary space of the colligation represents the "external world" which houses inputs and outputs which interact with the *internal states* coming from the original system. With the appropriate quadratic form on the *external space*, an energy conservation law is recovered for the enlarged more complicated system. The old spectral analysis for operator colligations described above then enables one to analyze how to decompose a given system into simpler systems, among other things.

2. *Nonstationary stochastic processes*. A stochastic process is simply a complex-valued measurable function $z = z(t, w)$: $\mathbf{R} \times \Omega \to \mathbf{C}$ where $\mathbf{R}$ is the real line and (Ω, μ) is a probability measure space. Thus $z(t) \equiv z(t, \cdot)$ is a *random variable* in probabilistic language; the prediction problem is to estimate $z(t + \tau)$ from a knowledge of $\{z(s) | s \leqslant t\}$. If the process is *stationary*, that is, the *correlation function* $v(t, s) = E\{z(t) \cdot z(s)\} = \langle z(t), z(s)\rangle_{L^2(\mu)}$ depends only on the difference of the arguments ($v(t, s) = v(t - s)$), then the operator U_τ: $z(t) \to z(t + \tau)$ extends by linearity and continuity to a unitary operator on the Hilbert space $L^2(\mu)$. If one assumes that the process is *linearly representable*, one can get a concrete Hilbert space model for the original process which then makes problems such as prediction tractable via function theory techniques. This has led to the highly successful theory of stationary stochastic processes of the last four decades. In their book the authors are interested in developing an analogous machinery for nonstationary processes. In this case the generator of the group U_τ above is non-self-adjoint, and the older methods for studying non-self-adjoint operators via colligations can be brought in to play a role. They introduce the *infinitesimal correlation function*

$$w(t, x) = -\frac{\partial}{\partial \tau} v(t + \tau, s + \tau)\Big|_{\tau = 0}$$

where $v(t, s)$ is the correlation function, and then define the *nonstationariness rank* of the process to be the maximal rank r ($0 \leqslant r \leqslant \infty$) of all quadratic forms

$$\sum_{j,k=1}^{n} w(t_j, t_k)\xi_j\bar{\xi}_k \qquad (-\infty < t_1 < t_2 < \cdots < t_n, n = 1, 2, \ldots).$$

The form of the correlation function is found and spectral resolutions are obtained for various classes of linearly representable dissipative processes of given rank r; the techniques come from the theory of characteristic functions and triangular models discussed above. The material in the book per se does not solve any problems (such as prediction) but appears ready to be milked for future applications as it becomes better known.

3. *Refinements*. The reader will also find interspersed throughout the book various refinements of the original notion of colligation. Thus for example, a colligation is defined for an n-tuple of non-self-adjoint operators (rather than for a single operator); the corresponding characteristic function becomes a function of several variables. These notions can then be applied to multivariate stochastic processes (parametrized by $\mathbf{R}^n$ rather than $\mathbf{R}$). In recent work Livsic [**20**] and Kravitsky [**15**] have established some connections between characteristic functions for *commutative* colligations and algebraic geometry, and Livsic [**21**] has obtained a triangular model for any two commuting finite-dimensional operators. This last result I think is rather impressive in view of how attempts to generalize the Sz.-Nagy-Foias theory to several variables always seem to get bogged down in pathologies of the function theory on the polydisk [**2**], [**8**]–[**10**], [**23**]. Also the authors give a notion of colligation involving Riemannian differential geometry and discuss colligations invariant with respect to a group of transformations.

The style is rather formal and the going a little heavy at times with strings of definitions and theorems; also the list of references would leave one quite unaware that there was much activity in American operator theory, but these are minor points. I think the book is well worth the attention of researchers in operator and systems theory, probability and statistics; what's more, by today's standards, the price is quite reasonable.

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Nonlinear differential equations, by S. Fucik and A. Kufner, Studies in Applied Mechanics, vol. 2, Elsevier, New York, 1980, 359 pp., $83.00.

Recently three closely guarded secrets of modern mathematics and science have been revealed. They are

(i) the understanding of genuine nonlinear phenomena lies at the heart of many important problems in diverse areas of knowledge;

(ii) these nonlinear phenomena can often be adequately described by studying systems of nonlinear differential equations;

(iii) there are simple systematic mathematical ideas and techniques that are adequate to treat broad classes of these nonlinear systems. Moreover, when such ideas do not exist, they are being keenly pursued world-wide by many researchers, young and old.

Thus, each day seems to bring additional insights and significant mathematical results connecting the three facts mentioned above. These results are attained not only by professional mathematicians, but also by mathematically trained scientists and engineers whose work forces them to solve these problems.

All those who love mathematics have cause to rejoice since many modern mathematical areas developed until now for their own sake (e.g. homotopy groups of spheres and simple Lie groups, abelian functions, singularity theory, and the differential geometry of connections) are absolutely essential for the understanding of key nonlinear problems of science. These problems in turn spawn new fruitful directions of depth and subtlety for mathematics and science. Hidden links between diverse mathematical areas are being revealed. In short, we are witnessing the making of a new mathematical and scientific revolution.

The book under review explains some known (functional analysis) methods for certain classes of boundary value problems for certain nonlinear differential equations. The authors limit themselves to nonlinear elliptic equations