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T. KAMBAYASHI

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Introduction to H_p spaces, by Paul Koosis, London Mathematical Society Lecture Note Series, No. 40, Cambridge University Press, Cambridge, New York, Melbourne, 1980, XV + 376 pp.

Suppose f is a function holomorphic on the open unit disk, Δ, of $\mathbf{C}$, and suppose $0 < p < \infty$. Then f is said to be in H_p if

$$\sup_{r<1}\left(\int_0^{2\pi}|f(re^{i\theta})|^p\frac{d\theta}{2\pi}\right)^{1/p} \equiv \|f\|_{H_p} < \infty.$$

H_∞ is the ring of bounded holomorphic functions on Δ and is endowed with the supremum norm. H_p spaces were first studied by Hardy (hence the terminology H_p) in 1915 and have remained an object of active study to this day. The past two decades have seen an enormous amount of work, and rather successful extensions of the H_p theory to $\mathbf{R}^n$, $\mathbf{C}^n$, and various other topological spaces have been made. The one dimensional theory remains interesting, however, for (at least) two reasons. Firstly, the existence of tools peculiar to one complex variable (e.g. conformal mappings and Blaschke products) makes life easier there than in higher dimensional spaces. Indeed, much of the current research in H_p spaces is devoted to finding analogues in $\mathbf{R}^n$ or $\mathbf{C}^n$ of theorems known in dimension one. Secondly, the space H_∞ has no known analogue in the $\mathbf{R}^n$ theory for $n \geq 2$, and the $\mathbf{C}^n$ theory seems to be extremely difficult when $n \geq 2$. An example of the latter phenomenon is the inner function problem: can there exist a nonconstant function f bounded and holomorphic on the unit ball of $\mathbf{C}^2$ such that f has radial limits of modulus one almost everywhere on the unit sphere? In one dimension such examples abound; $f(z) = z$ will do the job.

The book of Koosis gives an introduction to the one dimensional theory of H_p spaces. A good way to see what techniques must be developed in such a book comes from looking at three sample theorems proved within the past twenty years.

1. *The maximal ideal space of H_∞ is the closure* (*in the Gelfand topology*) *of* Δ.

2. $(\operatorname{Re} H_1(0))^* = BMO$.

3. *The unit ball of H_∞ is the norm closed convex hull of the Blaschke products.*

Theorem 1 is the corona theorem and is due to L. Carleson. The proof in Koosis' book is the recent one due to T. Wolff. Central to the proof is the concept of Carleson measures. For an interval I on $\Pi = \partial\Delta$ let θ_I denote the center of I. A complex measure μ is called a Carleson measure if

$$\sup_I \frac{1}{|I|}|\mu|\left(\left\{z \in \Delta : |z - \theta_I| < \frac{|I|}{2}\right\}\right) \equiv \|\mu\|_c < \infty.$$

An easy example of a Carleson measure is Lebesgue measure on the interval $(-1, 1)$. A useful fact about Carleson measures is that

$$\int_\Delta |f(z)|d|\mu|(z) < C_0\|f\|_{H_1}\|\mu\|_C$$

for all $f \in H_1$. This fact is most easily proved by considering maximal functions. For a point $e^{i\theta} \in \Pi$ the cone Γ_θ is defined by

$$\Gamma_\theta = \left\{z : |z| > \frac{1}{\sqrt{2}}, |\arg(z - e^{i\theta})| < \pi/4\right\} \cup \left\{|z| < \frac{1}{\sqrt{2}}\right\}.$$

For a function f defined on Δ the maximal function is defined by $f^*(e^{i\theta}) = \sup_{z\in\Gamma_\theta}|f(z)|$. The basic fact about this maximal function is that $\|f^*\|_{L_p} \leqslant C_0\|f\|_{H_p}$, $0 < p \leqslant \infty$.

The corona problem can be reduced via functional analysis to the following problem. Suppose $f_1, \ldots, f_n \in H_\infty$ and $\sup_j|f_j(z)| \geqslant \delta > 0$ for all z. Show there are functions $g_j \in H_\infty$ such that $\sum_{j=1}^N f_j g_j \equiv 1$. To produce the functions g_j, first consider the (nonholomorphic) functions $\phi_j = \overline{f_j}/\sum_{k=1}^N f_k\overline{f_k}$. Then $\sum_{j=1}^N f_j\phi_j \equiv 1$. By using the useful fact about Carleson measures and the duality relation $(H_1(0))^* = L_\infty/H_\infty$ one can show that the $\bar\partial$ problem $\bar\partial F = \phi_j\bar\partial\phi_k$ has a solution $a_{j,k}$ with L^∞ boundary values on Π. Then $g_j = \phi_j + \sum_{k=1}^N(a_{j,k} - a_{k,j})f_k$ is holomorphic because

$$\begin{aligned}\bar\partial g_j &= \bar\partial\phi_j + \sum_{k=1}^N \left(\phi_j\bar\partial\phi_k - \phi_k\bar\partial\phi_j\right)f_k \\ &= \bar\partial\phi_j\left(1 - \sum_{k=1}^N f_k\phi_k\right) + \phi_j\left(\bar\partial \sum_{k=1}^N f_k\phi_k\right) = 0 + 0.\end{aligned}$$

We also know that $g_j \in H_\infty$ because its boundary values are in L_∞. The functions g_j solve the corona problem because

$$\sum_{j=1}^N f_j g_j \equiv \sum_{j=1}^N f_j\phi_j \equiv 1.$$

Theorem 2 is C. Fefferman's famous theorem on the duality between the real part of H_1 and BMO. A function f is said to be in Re $H_1(0)$ if f is the real part of $F \in H_1$ and $F(0) = 0$. The norm of f is defined to be $\|F\|_{H_1}$.

If $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \in L_1$ the Hilbert transform of f is $H_f = -\sum_{n=-\infty,n\neq 0}^{\infty} i(\operatorname{sgn} n)a_n e^{in\theta}$, and $F \in H_1$ if and only if $F = f + iHf$ where $f, Hf \in L_1$. A locally integrable function ϕ on Π is in BMO if

$$\sup_I \frac{1}{|I|}\int_I|\phi - \phi_I|dx \equiv \|\phi\|_* < \infty,$$

where $\phi_I = \int_I \phi \, dx/|I|$. A basic fact about the Hilbert transform and BMO is that H maps BMO to BMO boundedly. A functional analysis argument shows that if $\Lambda \in (\operatorname{Re} H_1(0))^*$ then $\Lambda(f) = \int_\Pi f\phi \, d\theta/2\pi$ for some function ϕ of the form $\phi = u + Hv$, $u, v \in L_\infty$. Consequently, $(\operatorname{Re} H_1(0))^* \subset$ BMO. To prove the opposite inclusion it is enough to show that $\int_\Pi F\phi \, d\theta/2\pi$ is bounded whenever $F \in H_1(0)$ and $\phi \in$ BMO is real valued. Standard tricks with Blaschke products show that it is sufficient to treat the case where $F = g^2$, $g \in H_2$. In this case Green's theorem shows

$$\int_\Pi F\phi \frac{d\theta}{2\pi} = \frac{1}{\pi}\int_\Delta (\nabla F \cdot \nabla\phi)\log\frac{1}{|z|}\, dx\, dy$$

$$< \frac{2}{\pi}\left(\int_\Delta |g'|^2 \log\frac{1}{|z|}\, dx\, dy\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_\Delta |F|\,|\nabla\phi|^2 \log\frac{1}{|z|}\, dx\, dy\right)^{\frac{1}{2}} = \frac{2}{\pi} A \cdot B.$$

The Littlewood-Paley identity shows $A^2 < C_0\|g\|_{H_2} = C_0\|F\|_{H_1}$. Since $B^2 < C_0\|F\|_{H_1} |\nabla\phi|^2 \log 1/|z|\, dx\, dy\|_C$ the theorem follows from C. Fefferman's lemma: $\| \, |\nabla\phi|^2 \log 1/z \, dx\, dy\|_C \leq C_0\|\phi\|_*$.

Theorem 3 is due to D. Marshall. There are three crucial ingredients to the proof. The first is a (weak form of a) theorem that R. Nevanlinna proved in 1929. If $f \in H_\infty$, $\|f\|_{H_\infty} < 1$, and if $\{z_j\}$ is a sequence of points in Δ satisfying $\Sigma(1 - |z_j|) < \infty$, then there is an inner function U (i.e. $|U(e^{i\theta})| = 1$ a.e.), $U \in H_\infty$, and $U(z_j) = f(z_j)$ for all j. The second ingredient is a result due to Douglas and Rudin: If $V \in L^\infty$ and $|V(e^{i\theta})| = 1$ a.e. then given $\varepsilon > 0$ there are Blaschke products B_1 and B_2 such that $\|V - B_1/B_2\|_{L^\infty} < \varepsilon$. The third ingredient is "A. Bernard's trick": If A is a uniform algebra on a compact Hausdorff space Y and A is generated by $\mathcal{U} = \{U \in A: |U| \equiv 1$ on $Y\}$, then the unit ball of A is the norm closed convex hull of $\mathcal{U}$. These three results plus an ingenious concatenation due to Marshall now prove Theorem 3.

These then are a few of the things one must understand in order to follow current research in the H_p theory: Blaschke products and inner functions, maximal functions, Carleson measures, duality theorems, $\bar\partial$ problems, Hilbert transforms, Littlewood-Paley theory, BMO, and H_∞ interpolation. The book of Koosis provides a clear and thorough introduction to these topics as well as other basic ones such as boundary values, the F. and M. Riesz theorem, outer functions, and Beurling's theorem. The purpose and scope of Koosis' book are best conveyed by the following quote from the jacket cover. "The author's aim is to give the reader, assumed to know basic real and complex variable theory and a little functional analysis, enough background to be able to follow current research in the subject and its applications. For this reason, the emphasis is on methods and the ideas behind them rather than on the accumulation of as many results as possible. Computations are done in detail and many diagrams are provided throughout." Koosis has succeeded admirably in all that he set out to do in his book. In addition to the highly enjoyable writing style, Koosis' book contains 66 gorgeous diagrams and a sizable bibliography. If you want to learn about H_p spaces, here is an excellent place to start.

PETER W. JONES

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Number theory, by Helmut Hasse, Grundlehren der Mathematischen Wissenschaften, Band 229, A Series of Comprehensive Studies in Mathematics, Springer-Verlag, Berlin and New York, 1980, xvii + 638 pp., $49.00. A corrected and enlarged translation of the third edition of *Zahlentheorie*, Akademie-Verlag, Berlin, 1969, edited and prepared for publication by Horst Günter Zimmer.

The late Helmut Hasse wrote two treatises on number theory. Their first editions appeared in rapid succession in 1949 and 1950. The first, the "blue book", was entitled *Zahlentheorie* and was published by Akademie-Verlag (2nd ed. 1963, 3rd ed. 1969). It was a book on algebraic number theory. The second, the "yellow book", was *Vorlesungen über Zahlentheorie*, a book on elementary number theory published by Springer-Verlag (2nd ed. 1964).

Lovers of number theory will now have to be a little careful: both books are yellow. The volume under review is a translation into English of the third edition of the blue book; in moving from Akademie to Springer it changed color. Beyond that the major change is a recasting of Chapter 16 (on tamely ramified extensions) to remove an error detected by Leicht and Roquette; the rewriting was done by Leicht.

None of the earlier editions was reviewed in this Bulletin. I think a review is still timely, for it is a fine book. It treats algebraic number theory from the valuation-theoretic viewpoint. When it appeared in 1949 it was a pioneer. Now there are plenty of competing accounts. But Hasse has something extra to offer. This is not surprising, for it was he who inaugurated the local-global principle (universally called the Hasse principle). This doctrine asserts that one should first study a problem in algebraic number theory locally, that is, at the completion of a valuation. Then ask for a miracle: that global validity is equivalent to local validity. Hasse proved that miracles do happen in his five beautiful papers on quadratic forms of 1923–1924. But I cannot end this paragraph without calling attention to Hasse's eloquent attribution of the key idea to his teacher Hensel; see vol. 209 (1962) of Crelle and an amplification in his preface to volume I of his *Mathematische Abhandlungen*.

We can now read in English Hasse's disavowal of a "Satz-Beweis" format in favor of a discursive exposition. And indeed the exposition is discursive. The first 100 pages take the reader on a trip through elementary number theory that reaches quadratic reciprocity in the Hilbert product formula version. Then comes a 200-page book within a book on valuation theory. At last, halfway through the book, algebraic number theory begins. Examples