

topics. These are all carefully treated with the exception of base change. Perhaps I should explain, since the phrase has come up several times, that base change refers to the analogue in étale theory of the result in singular theory that for a proper map of reasonable topological spaces $f: Y \rightarrow X$, one has for $x \in X$

$$H^*(f^{-1}(x)) \cong \varinjlim H^*(f^{-1}(U))$$

where U runs through neighborhoods of x . (The proof follows from the existence of neighborhood retracts of $f^{-1}(x)$ in Y .) Given the importance of this result, Milne's treatment is much too rapid. I recommend instead the proof in Deligne's Springer Lecture Notes SGA 4 $\frac{1}{2}$.

In sum, the author has done a tremendous service by organizing the material in a careful and united way which makes it possible for serious students to learn. In its way, the terse and brilliant account of the theory by Deligne (who disposes of the whole business in 65 pages) in SGA 4 $\frac{1}{2}$ is unexcelled. On the other hand, having watched graduate students trying to make sense of the many details, I can testify to the need for a book like this. Having tried to teach a course in the subject, I can testify to the achievement it is to write one.

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Curves on rational and unirational surfaces, by M. Miyanishi, Tata Institute of Fundamental Research, Bombay, 1978, Narosa Publishing House, New Delhi, 1978, 307 pp., \$9.90.

Traditionally, algebraic geometry has meant the study of projective varieties, with its richest results having been produced for curves and surfaces in projective spaces. So, it was not always clear how deeply algebraic geometry was related to commutative algebra. Until rather recently, one might almost perfunctorily start out with affine varieties as zeros of ideals in a polynomial ring, but as soon as one got serious one would switch over to projective varieties and geometric arguments. Indeed, the great Italians (Castelnuovo, Enriques, Severi, . . .) appeared oblivious to commutative algebra while developing their immensely successful algebraic surface theory. Even Hilbert,

in his famous 1890 'theology' paper, concerned himself only with homogeneous ideals (essentially projective objects) when he was in fact proving a theorem about general ideals in a polynomial ring (viz. the basis theorem). Such preoccupation with things projective rather than affine is understandable because of the compactness of projective varieties ('proper over the ground field' as put today) and the resulting rigidity of their structure. By this last I mean that the automorphism group of a projective variety is reasonably small (viz. locally algebraic), in sharp contrast to the affine case.

Today's algebraic geometry has, of course, a much more varied outlook than studying just projective varieties. In particular, a substantial segment of research in this field now deals with affine varieties and especially affine spaces. (Here, by an affine variety is meant a closed algebraic subvariety of an affine n -space A_k^n over some ground field k ; the affine space A_k^n is, technically, the prime spectrum $\text{Spec}(k[T_1, \dots, T_n])$ of the polynomial ring over k and, intuitively, just \bar{k}^n (\bar{k} = alg. closure of k .) Problems and results in this area pertain to complete intersection and embedding in A^n , fibrations by A^n , forms of A^n relative to Grothendieck topologies, A^n -bundles and vector bundles, and many others. The author of the book under review has been a leader in this area. When he was invited by Tata Institute of Fundamental Research in Bombay to deliver a series of lectures in early 1978, he prepared beforehand a set of notes for his own use summarizing various recent results of his own as well as others' concerning the geometry of A^n . The present book reproduces these lecture notes with some redaction put in afterwards. Incidentally, the author told me the following episode: When he announced, at the beginning of his lecture series at Tata, that he would be looking into geometry of the affine line A^1 , many in the audience laughed at what they apparently thought was a joke. Surely, A^1 is like C , for instance, when the ground field is the complex number field C , so what geometry is there to be looked into?!

Before going into particulars of the book, let me lay out some more background material about algebraic geometry of the affine space or, equivalently, that of the polynomial ring. First, let me clarify these phrases as referring to that branch of affine algebraic geometry which deals directly with affine spaces and polynomial rings; whereas affine algebraic geometry is the study of general affine varieties leading up to their ultimate classification. In this reviewer's view, algebraic geometry of the affine space was born sometime around 1970. By then, Grothendieck's *Éléments* [10] had become the orthodoxy and there was no longer any doubt that a substantial part of commutative algebra is precisely equal to affine algebraic geometry. A practical implication of this recognition was that a good number of talented mathematicians who had started out as commutative algebraists, K -theorists and the like now got interested in geometric problems and began producing excellent results of strongly geometric flavor. Also by 1970, the proverbial pendulum had begun to swing the other way: trend of research in algebraic geometry had turned from the abstract to the concrete. Thus, one was now more apt to study lines and planes in A^3 , vector bundles on P^3 , and $K3$ surfaces rather than, say, spectral sequence in étale cohomology for motifs.

So, beginning around 1970, an increasing number of algebraic geometers and commutative algebraists joined the ranks of the students of the affine space and the polynomial ring. To cite just a few of the publications at this early stage of the decade which helped stimulate interest in this direction, there are Abhyankar's lecture notes [1] and Nagata's [26]; Russell's inspiring work [29] on forms of A^1 and G_a ; Ramanujam's paper [28] on the affine plane; first results on cancellation questions as in Abhyankar-Eakin-Heinzer [2] and Hochster [11], and Eakin-Heinzer's report [6] on the subject; and progress made on complete-intersection questions by Murthy [25], Storch [31] and Eisenbud-Evans [7].

During the 1970's three major results were obtained in the geometry of A^n ; namely, in chronological order:

(A) Abhyankar-Moh's Embedding Theorem [3] about lines in A^2 ;

(B) Quillen-Suslin's solution [27], [32] of Serre's Problem about vector bundles on A^n ;

(C) Fujita-Miyayishi-Sugie's solution [8], [20], [23] of the cancellation problem for A^2 .

It is interesting to observe that, while these three results may have established the geometry of A^n as a viable branch in algebraic geometry, ironically the methods employed for them have proved once again how powerful the techniques of projective algebraic geometry are. For, the essence of all proofs of (A) so far available is to embed the affine plane A^2 containing a curve $C \simeq A^1$ into the projective plane P^2 and to analyze the singularity that occurs to C at infinity; proof of (B) was initially attained by arguments involving vector bundles on the projective n -space P^n (although this has turned out to be avoidable); and the result (C) relies in essential manners upon the more classical theory of projective rational surfaces.

While the observation just made about the classicity of methods is very true, there is one other important factor which should not be overlooked. Namely, the role played in the solution (C) by the new algebraic geometry of open varieties developed by S. Iitaka [12], [13] and his school. In fact, the solution of the cancellation problem (to show that if $X \times A^1 \simeq A^3$ then $X \simeq A^2$) was achieved only when Fujita [8] completed Miyayishi-Sugie's work [23] and proved that an affine surface with logarithmic Kodaira dimension $-\infty$ is ruled in the affine sense, i.e., it contains a 'cylinderlike' open set (cf. [19], [30] also).

Miyayishi's lecture notes under review were written before the result (C) was established. In fact, as the author states in the opening paragraphs, his main motivation in writing these notes was to put together the results obtained up until then surrounding the cancellation problem. So, the notes naturally do not contain the latest solution of the problem or its refinements. On the other hand, a full account of the theorem (A) and all allied results is contained in this volume.

Let us now go into some of the details of the book:

In the main, Chapter I deals with affine surfaces containing base-point-free pencils of affine lines, i.e., containing 'cylinderlike' Zariski open subsets isomorphic to $C \times_k A^1$, product of a curve C and the affine line A^1 over an

algebraically closed ground field k . Conditions as to when a surface contains such a pencil are discussed in §1 in terms of derivations and actions of the additive group G_a . A main result in the chapter is the *characterization theorem for the affine plane* (Miyanishi [20]):

THEOREM. *Let $X = \text{Spec } A$ be an affine surface over an algebraically closed field k . Assume that (i) A is factorial (a UFD), (ii) $A^* = k^*$ (i.e., no nontrivial units exist), (iii) there exists a nontrivial G_a -action on X . Then, $X \simeq \mathbb{A}^2$ (and conversely).*

It is implicit in the author's discussion of this theorem but became clearly recognized later (Miyanishi-Sugie [23]) that the condition (iii) may be replaced by (iii*): X contains a cylinderlike open set $C \times \mathbb{A}^1$. Today, we have a very elegant, purely algebraic proof of ((i) + (ii) + (iii*)) $\Leftrightarrow X \simeq \mathbb{A}^2$ due to R. G. Swan (unpublished) as well as other proofs (see, e.g., [19]).

Other significant results discussed in Chapter I include two theorems due to the author and the reviewer [16] concerning flat fibrations by the affine line \mathbb{A}^1 . The first theorem says that, if the generic fiber of a flat, finite-type morphism $f: X \rightarrow S$ is \mathbb{A}^1 and all other fibers are geometrically integral, then X is an \mathbb{A}^1 -bundle over S . (This last means S may be covered by Zariski open sets $\{U_i\}$ so that $f^{-1}(U_i) \simeq U_i \times \mathbb{A}^1$ for all i .) The base scheme S is assumed to be locally factorial here.¹ The second theorem is more geometric in set-up and content and asserts, roughly, that a flat family of curves whose general members are \mathbb{A}^1 's contains a cylinderlike subfamily $U \times \mathbb{A}^1$. (If $\text{char}(k) \neq 0$, a purely inseparable, finite base extension may have to be performed first.) The chapter ends with a very interesting classification of \mathbb{A}^1 -bundles over \mathbb{P}^1 and a discussion of the cancellation problem.

Chapter II is clearly the most innovative part of the book. Here, one is to deal with a smooth affine rational surface containing a pencil of rational curves which carry only one-place singular points at infinity. The surface may be embedded in a smooth projective surface so that the initial affine pencil yields a pencil of rational curves in the more traditional, projective sense. The important work to be done in this situation is to scrutinize how the base points and the singularities of the pencil at infinity are resolved through a series of blowing-ups. Abhyankar and Moh were the first to recognize the usefulness of such scrutiny and conducted it with their own device of approximate roots (Tschirnhausen transformation) of a polynomial. As a reward, they obtained the very beautiful Embedding Theorem and other theorems closely connected with it [3], [4]. In the lecture notes at hand, the author constructs his own proof of Abhyankar-Moh's results, aided by the notion of *admissible datum* due to himself. After proving a few technical results about admissible data which describe curves with one-place points at infinity on smooth rational surfaces, Miyanishi is able to recover, with a few extra strokes, all the major results in this vein: Irreducibility Theorem (Moh [24]); Generic Irreducibility Theorem (Ganong [9]); Embedding and Epimorphism Theorems (Abhyankar-Moh [3], Abhyankar [4]); and Automorphism

¹V. Danilov has an unpublished proof which works when S is assumed to be normal.

Theorem (van der Kulk [33]). To indicate the flavor of these important theorems, let me state two of them:

EMBEDDING THEOREM. *Let $C: f(x, y) = 0$ be a curve in \mathbf{A}^2 isomorphic to \mathbf{A}^1 . In case $\text{char}(k) = p > 0$, assume additionally that $p \nmid$ (total degree of f) or $p \nmid$ (multiplicity of C at ∞). Then, there exists $g \in k[x, y]$ such that $k[f, g] = k[x, y]$. (Namely, C may be taken as a coordinate axis.)*

AUTOMORPHISM THEOREM. *Let $G := \text{Aut}_k k[x, y]$ be the automorphism group of \mathbf{A}_k^2 . Then, G is a free product of A_2 and J_2 amalgamated along $A_2 \cap J_2$, where A_2 is the subgroup of affine transformations $(x \mapsto ax + by + c, y \mapsto a'x + b'y + c')$ and J_2 is that of de Jonquières transformations $(x \mapsto ax + c, y \mapsto b'y + P(x))$ ($P(x) \in k[x]$).*

Van der Kulk's Automorphism Theorem just stated was proved in 1953, but remained in obscurity until the 1970's.² It was then taken up and reproved by Abhyankar-Moh and Nagata [26], and was applied by the reviewer to show the absence of separable forms of \mathbf{A}^2 and to determine all algebraic group actions on \mathbf{A}^2 (Kambayashi [15], [18]).

In the remainder of Chapter II the author discusses the question of embedding curves of positive genus in \mathbf{A}^2 , simple birational extensions of $k[x, y]$, and affine plane curves with two places at infinity. The second topic was studied by the author, Russell, Sathaye and others in connection with the cancellation problem for \mathbf{A}^2 (now completely settled), while the third topic is related to the Jacobian Problem about polynomial endomorphisms of \mathbf{A}^2 (still an open and important question).

The author shifts the gear in Chapter III and treats complete smooth surfaces directly rather than uses them as tools to study affine objects. He is here concerned with quasi-elliptic surfaces. In the classification of complete smooth surfaces, the quasi-elliptic surfaces occur as sort of anomalies when the ground field characteristic is 2 or 3. (Consult [5] for general background and bibliography.) The starting point of Miyanishi's work in this area is the observation that the generic fiber of a quasi-elliptic surface $S \rightarrow C$, minus its unique singular point, is a form of the affine line \mathbf{A}^1 over the function field $K := k(C)$ of C . Here, by a form of \mathbf{A}^1 over K is meant an algebraic variety X defined over K such that after base extension $X \otimes_K K'$ is K' -isomorphic to \mathbf{A}^1 for some algebraic extension field $K' \supseteq K$. When that is so, X must be a smooth affine curve and K' can be chosen to be a finite, purely inseparable extension of K . Forms of \mathbf{A}^1 were found by M. Rosenlicht, and were systematically studied by Russell [29]. Their work was greatly expanded by Miyanishi, Takeuchi and the reviewer [14], [17], and among other things all forms of \mathbf{A}^1 of genus one were determined in [14, §6]. Using this last result and the extra assumption that S be unirational (equivalently, C be rational) and there exists a cross-section to $S \rightarrow C$, the author is able to describe such

²H. W. E. Jung had shown in 1942 that $\text{Aut}(\mathbf{A}^2)$ was generated by A_2 and J_2 when the ground field was \mathbf{C} (Crelles Journal, vol. 184).

quasi-elliptic surfaces explicitly.³ For detailed statements, the reader is referred to the main text (Chapter III, §2) or to Miyanishi's original papers [21], [22].

For the most part the techniques employed in this book are those of classical (rational) surface theory involving canonical divisors, blowing-ups, intersection theory and so on. (There are, however, some exceptions in Chapter I.) One marvels at the variety and the richness of results as found in this volume obtained by means of such limited techniques. At the same time, one must remember that Miyanishi was unable to solve the cancellation problem ($X \times \mathbf{A}^1 \simeq \mathbf{A}^3 \Rightarrow X \simeq \mathbf{A}^2$?) within the scope of the classical, complete surface theory methods. As noted already, he had to find help from Iitaka's modern theory of open varieties and their logarithmic invariants [12], not to mention Fujita's forceful arguments [8]. In pondering over this situation, I wonder very much about two things:

(1) The Jacobian problem asks if a polynomial endomorphism of \mathbf{C}^2 (\mathbf{C} = the complex number field) whose Jacobian determinant equals 1 is an automorphism. This question, which motivated many into geometry of \mathbf{A}^2 in the early 1970's, remains unsolved in 1980. Does this suggest that an entirely new, powerful technique must be found before this problem is resolved? Unfortunately, it seems that no one has yet found ways to apply Iitaka's theory effectively to this problem.

(2) If we are to deal successfully with problems about the affine 3-space \mathbf{A}^3 or more generally about \mathbf{A}^n ($n > 2$), must we wait for a sophisticated theory of complete smooth threefolds (= algebraic varieties of dimension 3) to be first developed? Until such time comes, is it hopeless even to try? For example, could we clearly understand the structure of $\text{Aut}(\mathbf{A}^3)$ without knowing all about different compactifications of \mathbf{A}^3 ? Perhaps the optimist would find encouragement in the way Serre's Problem ((B) above) was solved without any inherently low-dimensional, projective-geometric techniques.

Miyanishi's 1978 lecture notes are already somewhat out-of-date, especially in connection with the cancellation problem. (In fact, the author himself contributed to this obsolescence!) Still, the book is an excellent illustration of how far one can go in the application of classical methods to the geometry of affine lines and surfaces. I recommend the book to young students of algebraic geometry.

In closing, let me add that Miyanishi has prepared a set of notes entitled *Classification theory of noncomplete algebraic surfaces* for his lectures to be delivered at the University of Chicago in the Autumn Quarter of 1980. These latest notes, about 250 pages, will be published before long in Springer-Verlag's *Lecture Notes in Mathematics* series and should be a natural sequel to the present ones under review.

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Introduction to H_p spaces, by Paul Koosis, London Mathematical Society Lecture Note Series, No. 40, Cambridge University Press, Cambridge, New York, Melbourne, 1980, XV + 376 pp.

Suppose f is a function holomorphic on the open unit disk, Δ , of \mathbf{C} , and suppose $0 < p < \infty$. Then f is said to be in H_p if

$$\sup_{r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} \equiv \|f\|_{H_p} < \infty.$$

H_∞ is the ring of bounded holomorphic functions on Δ and is endowed with the supremum norm. H_p spaces were first studied by Hardy (hence the terminology H_p) in 1915 and have remained an object of active study to this day. The past two decades have seen an enormous amount of work, and rather successful extensions of the H_p theory to \mathbf{R}^n , \mathbf{C}^n , and various other topological spaces have been made. The one dimensional theory remains interesting, however, for (at least) two reasons. Firstly, the existence of tools peculiar to one complex variable (e.g. conformal mappings and Blaschke products) makes life easier there than in higher dimensional spaces. Indeed, much of the current research in H_p spaces is devoted to finding analogues in \mathbf{R}^n or \mathbf{C}^n of theorems known in dimension one. Secondly, the space H_∞ has no known analogue in the \mathbf{R}^n theory for $n > 2$, and the \mathbf{C}^n theory seems to be extremely difficult when $n > 2$. An example of the latter phenomenon is the inner function problem: can there exist a nonconstant function f bounded and holomorphic on the unit ball of \mathbf{C}^2 such that f has radial limits of modulus one almost everywhere on the unit sphere? In one dimension such examples abound; $f(z) = z$ will do the job.

The book of Koosis gives an introduction to the one dimensional theory of H_p spaces. A good way to see what techniques must be developed in such a book comes from looking at three sample theorems proved within the past twenty years.

1. *The maximal ideal space of H_∞ is the closure (in the Gelfand topology) of Δ .*

2. *$(\operatorname{Re} H_1(0))^* = BMO$.*

3. *The unit ball of H_∞ is the norm closed convex hull of the Blaschke products.*