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RESEARCH ANNOUNCEMENTS

THE TETRAGONAL CONSTRUCTION

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1. **Preliminaries.** Let C be a nonsingular curve of genus g, and $\pi: \widetilde{C} \longrightarrow C$ an unramified double cover. The Prym variety $P(C, \widetilde{C})$ is by definition $\ker^0(Nm)$, where $Nm: J(\widetilde{C}) \longrightarrow J(C)$ is the norm map, and $\ker^0$ is the connected component of 0 in the kernel. By **[M]** this is a $(g - 1)$-dimensional, principally polarized abelian variety. Let A_g, M_g, R_g denote, respectively, the moduli spaces of g-dimensional principally polarized abelian varieties, curves of genus g, and pairs $(C, \widetilde{C})$ as above. (R_g is a $(2^{2g} - 1)$-sheeted cover of M_g.) The Prym map is the morphism

$$P = P_g: R_g \longrightarrow A_{g-1}, \qquad (C, \widetilde{C}) \mapsto P(C, \widetilde{C}).$$

It is analogous to the Jacobi map $J = J_g: M_g \longrightarrow A_g$ sending a curve to its Jacobian. The main reason for studying P is that its image in R_{g-1} is larger than that of J, hence it allows us to handle geometrically a wider class of abelian varieties than just Jacobians. For instance, P_g is dominant for $g \leqslant 6$ **[W]** while J_g is only dominant for $g \leqslant 3$.

The purpose of this announcement is to describe the fibers of P in the various genera. Our main tool for this is a simple-minded construction which we describe in some detail in paragraph 6. Let us use "n-gonal" (trigonal, tetragonal, etc.) to describe a pair (C, f) where $f: C \longrightarrow \mathbf{P}^1$ is a branched cover of degree n (3, 4 respectively). Briefly, our construction takes the data $(C, \widetilde{C}, f)$ where $(C, \widetilde{C}) \in R_g$ and (C, f) is tetragonal, and returns two new sets of data, $(C_0, \widetilde{C}_0, f_0)$ and $(C_1, \widetilde{C}_1, f_1)$, of the same type. This procedure is symmetric: starting with $(C_0, \widetilde{C}_0, f_0)$ we end up with $(C, \widetilde{C}, f)$ and $(C_1, \widetilde{C}_1, f_1)$. It is useful due to the following observation.

PROPOSITION 1.1. *The tetragonal construction commutes with the Prym map*:

$$P(C, \widetilde{C}) \approx P(C_0, \widetilde{C}_0) \approx P(C_1, \widetilde{C}_1).$$

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REMARK 1.2. A similar construction was studied by Recillas **[R]**, **[DS, III]**. He starts with a tetragonal pair (C, f) and produces a triplet $(X, \widetilde{X}, g)$ where (X, g) is trigonal and $\mathcal{P}(X, \widetilde{X}) \approx \mathcal{J}(C)$. This becomes the special case of our construction where $\widetilde{C}$ is taken to be the split double cover of C. The resulting C_0, C_1 are then isomorphic to X with a $\mathbf{P}^1$ attached (in two different ways) and

$$\mathcal{P}(C_i, \widetilde{C}_i) \approx \mathcal{P}(X, \widetilde{X}) \approx \mathcal{J}(C) \approx \mathcal{P}(C, \widetilde{C}).$$

2. Genus 6. In **[DS]** the map $\mathcal{P}: \mathcal{R}_6 \longrightarrow \mathcal{A}_5$ was studied at length. The main result was that this map is generically finite, of degree 27.

THEOREM 2.1. *The fibers of* $\mathcal{P}: \mathcal{R}_6 \longrightarrow \mathcal{A}_5$ *have a structure equivalent to the intersection-configuration of the* 27 *lines on a cubic surface.*

An equivalent formulation is

COROLLARY 2.2. *The Galois group of the field extension* $K(\mathcal{A}_5) \subset K(\mathcal{R}_6)$ *is the Weyl group* $W(E_6)$. (*Compare* **[Ma**, *Theorem* 23.9**]**).

The theorem limits severely the possible degenerations in a fiber of $\mathcal{P}$. For instance

COROLLARY 2.3. *The ramification locus* (*in* $\mathcal{R}_6$) *is mapped six-to-one to the branch locus* (*in* $\mathcal{A}_5$).

PROOF. A line on a cubic surface S counts twice if and only if it passes through a double point of S. Through such a point there are six lines. □

The proof of the theorem depends on the existence of 5 tetragonal maps, f_i $(1 \leqslant i \leqslant 5)$ on a generic curve C of genus 6. To each triplet $(C, \widetilde{C}, f_i)$ the tetragonal construction associates two others; the ten resulting points of $\mathcal{P}^{-1}\mathcal{P}(C, \widetilde{C})$ are the ones "incident" to $(C, \widetilde{C})$.

The same method allows us to recover the main result of **[DS]** rather painlessly: we show that starting with $(C, \widetilde{C}) \in \mathcal{R}_6$, choosing a tetragonal f, applying the tetragonal construction to get $(C_0, \widetilde{C}_0, f_0)$, changing the tetragonal f_0 to an f_0' and repeating the process indefinitely, leads to precisely 27 distinct objects: to the original $(C, \widetilde{C})$ are added ten after the first cycle, and only sixteen more after the second cycle. (I.e. each of the five first-generation pairs yields the *same* set of sixteen second-generation objects!) Therefore $\deg(\mathcal{P})$ is a multiple of 27. This possible multiplicity is eliminated by checking a degenerate case, where C is a double cover (branched) of an elliptic curve ("elliptic hyperelliptic").

3. Genus 5. The map $\mathcal{P}_5: \mathcal{R}_5 \longrightarrow \mathcal{A}_4$ turns out, surprisingly, to be more intricate than its higher-genus cousin $\mathcal{P}_6$, and until now has eluded description.

By dimension count, the generic fiber is 2 dimensional; we show that in fact it is a double cover of a Fano surface.

THEOREM 3.1. *There is a birational isomorphism* $\kappa: \mathcal{A}_4 \longrightarrow \mathcal{C}$ *where* $\mathcal{C}$ *is a parameter-space for pairs* (X, μ) *consisting of (the isomorphism class of) a cubic threefold* X *together with an "even" point of order two in its intermediate Jacobian.*

PROPOSITION 3.2. *There is a natural involution* $\lambda: \mathcal{R}_5 \longrightarrow \mathcal{R}_5$ *such that* $\lambda(C, \widetilde{C})$ *is related to* $(C, \widetilde{C})$ *by a succession of two tetragonal constructions; hence* $\mathcal{P} \circ \lambda = \mathcal{P}$.

THEOREM 3.3. *For generic* $A \in \mathcal{A}_4$, *the quotient* $\mathcal{P}^{-1}(A)/\lambda$ *is isomorphic to* $\mathcal{F}(\kappa(A))$, *the Fano surface of lines on the cubic threefold* $\kappa(A)$

The proofs seem to depend heavily on the results for genus 6 and their various specializations. As a corollary, we have an explicit parametrization of the family of (rational equivalence classes of) effective symmetric representatives of the class $[\theta]^3/3$ in $H_2(A, \mathbf{Z})$. This is twice the class of a curve in its Jacobian, and the smallest class which is effective on generic A.

4. Prym-Torelli. For $g \leqslant 4$ the analysis of $\mathcal{P}_g$ is fairly easy. It can be done using nothing but Recillas' trigonal construction (1.2), since any $A \in \mathcal{A}_{g-1}$ is Jacobian of a tetragonal curve. In the remaining cases $g \geqslant 7$, $\mathcal{P}_g$ "ought" to be injective by dimension count. After some inconclusive work of Tjurin **[T]**, counterexamples to this expected Prym-Torelli theorem were exhibited by Beauville **[B$_2$]** for $g \leqslant 10$, using Recillas' construction applied to curves which are tetragonal in two distinct ways. Using the tetragonal construction we exhibit counterexamples for all g. Without much justification we make the following

CONJECTURE 4.1. *If* $\mathcal{P}(C, \widetilde{C}) \approx \mathcal{P}(C', \widetilde{C}')$ *then* $(C', \widetilde{C}')$ *is obtained from* $(C, \widetilde{C})$ *by successive applications of the tetragonal construction. In particular,* C *and* C' *are tetragonal curves.*

5. Andreotti-Mayer varieties. In **[AM]**, Andreotti and Mayer studied the Schottky problem of characterizing Jacobians among abelian varieties. Call $A \in \mathcal{A}_g$ an $A - M$ variety if its theta divisor θ has a $(g-4)$-dimensional singular locus, and let $\mathcal{N}_g \subset \mathcal{A}_g$ be the closure of the locus of $A - M$ varieties. The main results of **[AM]** are that $\mathcal{N}_g$ can be explicitly described by equations, and that $\overline{\mathcal{J}(\mathcal{M}_g)}$ is an irreducible component of $\mathcal{N}_g$. Perhaps the most spectacular application of Prym theory was Beauville's refinement of their results **[B1]**. He obtained a complete (and lengthy) list of all possible components of $\mathcal{P}^{-1}(\mathcal{N}_g)$, hence, in principle, a description of $\mathcal{N}_4$, $\mathcal{N}_5$ (since $\mathcal{P}_5$, $\mathcal{P}_6$ are surjective, when appropriately compactified). In particular, he showed that $\mathcal{N}_4$ has only one irreducible component other than $\mathcal{J}(\mathcal{M}_4)$.

Using the tetragonal construction, some remarkable coincidences appear in Beauville's list. In fact

THEOREM 5.1. (1) *$\mathcal{N}_4$ consists of $\mathcal{J}_4$ and another nine-dimensional irreducible component* [**B1**].

(2) *$\mathcal{N}_5$ consists of $\mathcal{J}_5$ and four irreducible, nine-dimensional loci; three of these parametrize Pryms of elliptic-hyperelliptic curves, and the fourth consists of certain abelian varieties isogenous to a product with an elliptic curve.*

(3) *For $g \geqslant 6$, $\mathcal{N}_g \cap \overline{\mathcal{P}(\mathcal{R}_{g+1})}$ consists of $\mathcal{J}_g$, 2 components of Pryms of elliptic-hyperelliptic curves (each $(2g-1)$ dimensional) and $[(g-2)/2]$ components of Pryms of reducible curves $C = C_1 \cup C_2$, $\#(C_1 \cap C_2) = 4$ (each $(3g-4)$ dimensional).*

COROLLARY 5.2. *Any $(C, \widetilde{C}) \in \mathcal{P}^{-1}(\mathcal{N}_g)$ is either tetragonal (or a degeneration of tetragonals) or reducible. The modified Prym-Torelli Conjecture* 4.1 *holds over $\mathcal{N}_g$.*

CONJECTURE 5.3. *$\mathcal{N}_g \subset \overline{\mathcal{P}(\mathcal{R}_{g+1})}$, hence $\mathcal{N}_g$ consists only of the components listed above.*

The proof might imitate Andreotti's proof of Torelli's theorem and resurrect Tjurin's work [**T**]: Given $A \in \mathcal{N}_g$, there should be some explicit geometric construction yielding a family of doubly covered tetragonal (or reducible) curves, whose Prym is A.

COROLLARY 5.4. *For any canonical curve $C \subset \mathbf{P}^{g-1}$, the system of quadrics containing C is spanned by quadrics of rank* 4.

PROOF. A refinement of [**AM**] shows that the truth of the corollary for a given C depends only on the structure of $\mathcal{N}_g$ near $\mathcal{J}(C)$; in particular the corollary holds if $\mathcal{J}_g$ is the only component of $\mathcal{N}_g$ containing $\mathcal{J}(C)$. By Conjecture 5.3 and Theorem 5.1, this holds for all C except for hyperelliptics and elliptic-hyperelliptics. A special argument works for these.

6. The construction. We sketch the tetragonal construction. Start with an unramified double cover $\pi: \widetilde{C} \longrightarrow C$ and tetragonal map $f: C \longrightarrow \mathbf{P}^1$. Let

$$f_*(\pi): f_*(\widetilde{C}) \longrightarrow \mathbf{P}^1$$

be the "pushforward" of $\pi: \widetilde{C} \longrightarrow C$ via f. This is a $(16 = 2^4)$-sheeted branched cover. Over $p \in \mathbf{P}^1$, its 16 points correspond to the 16 ways of lifting the quadruple $f^{-1}(p) \subset C$ to a quadruple in $\widetilde{C}$. This suggests a convenient way of realizing $f_*(\widetilde{C})$ as a curve in $\mathrm{Pic}^{(4)}(\widetilde{C})$, the Picard variety of line bundles of degree 4: $f_*(\widetilde{C})$ is the subvariety parametrizing those effective divisors in $\widetilde{C}$ whose norm (under $\pi\colon \widetilde{C} \longrightarrow C$) is in the 1-dimensional linear series determined on C by f.

Note that on the curve $f_*(\widetilde{C})$ there is a natural involution $\tau\colon f_*(\widetilde{C}) \longrightarrow f_*(\widetilde{C})$. τ sends a lifting of $f^{-1}(p)$ to the complementary lifting, obtained by interchanging the sheets of $\pi\colon \widetilde{C} \longrightarrow C$. (This is induced by the automorphism of $\mathrm{Pic}^4(\widetilde{C})$ sending a line bundle L to $L^{-1} \otimes (f \circ \pi)^*\mathcal{O}_{\mathbf{P}^1}(1)$.) Let $\overline{C}$ be the quotient $f_*(\widetilde{C})/\tau$, an 8-sheeted cover of $\mathbf{P}^1$.

LEMMA. *$\overline{C}$ is reducible: $\overline{C} = C_0 \cup C_1$, each C_i is a 4-sheeted branched cover of $\mathbf{P}^1$. Correspondingly, $f_*(\widetilde{C}) = \widetilde{C}_0 \cup \widetilde{C}_1$, where $\widetilde{C}_i$ is acted upon by τ with quotient C_i.*

PROOF. Define an equivalence relation $\sim$ on $f_*(\widetilde{C})$: $D_1 \sim D_2$ if $f_*(\pi)(D_1) = f_*(\pi)(D_2)$ and D_1, D_2 have an even number of points (0, 2 or 4) in common. The quotient $f_*(\widetilde{C})/\sim$ is a 2-sheeted branched cover of $\mathbf{P}^1$. Clearly it can be branched only where $f\colon C \longrightarrow \mathbf{P}^1$ is; but a simple monodromy check shows that at such a point $f_*(\widetilde{C})/\sim$ is locally reducible. (I.e. in going around a branch point, an even number of points of $\widetilde{C}$ are exchanged.) Hence the normalization of $f_*(\widetilde{C})/\sim$ is nowhere ramified over $\mathbf{P}^1$, hence consists of two disjoint copies, so $f_*(\widetilde{C})$ itself is reducible. Finally, τ acts on each component separately since it changes an even number (all 4) of the points. Q.E.D.

Note. Identifying $\mathrm{Pic}^4(\widetilde{C}) \approx \mathrm{Jac}(\widetilde{C})$, we have that $f_*(\widetilde{C})$ is contained in the kernel of the norm-homomorphism, which [M] consists of two copies of the Prym variety; $\widetilde{C}_i$ are the intersections with these two components.

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