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STABLE AND L^2-COHOMOLOGY OF ARITHMETIC GROUPS

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Introduction. In [1], [2] we gave a range of dimensions in which the real cohomology of an arithmetic or S-arithmetic subgroup Γ of a connected semi-simple group G over $\mathbf{Q}$ is naturally isomorphic to the space of harmonic forms on the quotient $X = G(\mathbf{R})/K$ of the group $G(\mathbf{R})$ of real points of G by a maximal compact subgroup K which are invariant under Γ and the identity component $G(\mathbf{R})^0$ of $G(\mathbf{R})$, and indicated some applications to the stable cohomology of classical arithmetic groups and to algebraic K-theory. In this note we first state an extension to nontrivial coefficients, since this has become of interest in topology and K-theory [7]. A chief tool in [2] was the proof that $H^*(\Gamma; \mathbf{C})$ could be computed using differential forms on $\Gamma\backslash X$ which have "logarithmic growth" at infinity. Theorem 2 extends this to more general growth conditions. This can be used to show that certain L^2-harmonic forms are not cohomologous to zero [9]. In §§3, 4, 5 we consider the L^2-cohomology space $H_{(2)}(\Gamma\backslash X)$ and relate it to the spectral decomposition of the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$. Theorem 4 gives a sufficient condition under which it is finite dimensional, hence isomorphic to the space of square integrable harmonic forms, and §5 a series of examples in which it is not. For convenience, we assume G simple over $\mathbf{Q}$ and Γ torsion-free.

1. Let P_0 be a minimal parabolic $\mathbf{Q}$-subgroup of G, S a maximal $\mathbf{Q}$-split torus of P_0, N the unipotent radical of P and $\mathfrak{n}$ the Lie algebra of N. Let $X(S)$ be the group of rational characters of S and $\rho \in X(S)$ be such that $a^{2\rho} = \det \operatorname{Ad} a|_{\mathfrak{n}}$ for $a \in S$. For $\mu \in X(S)$ let $c(G, \mu)$ be the maximum of q such that $\rho - \mu - \eta > 0$, where η runs through the weights of S in $\Lambda^q\mathfrak{n}$. Let $c(G) = c(G, 0)$. If (r, E) is a finite-dimensional complex representation of $G(\mathbf{C})$, we let $c(G, r)$ be the minimum of $c(G, \mu)$, where μ runs through the weights of r with respect to S. It is easily seen that $c(G) \geqslant \Sigma_i c(G_i)$, where G_i runs through the simple factors of $G(\mathbf{C})$, and $c(G_i)$-is defined similarly, and that $c(G_i)$ is equal to $[(l-1)/2], l-1, l-2, l-1, 7, 13, 25, 5, 1$ if G_i is of type $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$.

THEOREM 1. *The natural homomorphism* $H^*(\mathfrak{g}, \mathfrak{k}; E)^\Gamma \longrightarrow H^q(\Gamma; E)$ *is injective for* $q \leqslant c(G, r)$, *surjective if in addition* $q < \operatorname{rk}_{\mathbf{R}} G$. *If* $E^G = (0)$, *then* $H^q(\Gamma; E) = 0$ *for* $q \leqslant c(G, r)$, $(\operatorname{rk}_{\mathbf{R}} G - 1)$. *If G is simply connected, these*

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assertions remain true if Γ is replaced by an S-arithmetic subgroup or by $G(\mathbf{Q})$.

Here $\mathfrak{g}$ and $\mathfrak{k}$ stand for the Lie algebras of $G(\mathbf{R})$ and K. See [5] for relative Lie algebra cohomology. The proofs of these statements are similar in principle to those given or sketched in [1], [2] when r is the trivial representation, and moreover, take into account some results proved in [5]. If we have an inductive system of groups and representations without trivial constituents $(G_n, \Gamma_n, r_n, E_n)$ such as (G, Γ, r, E) and if $c(G_n, r_n) \longrightarrow \infty$, then Theorem 1 implies that $H^q(\lim \Gamma_n, \lim E_n) = 0$ for $q > 0$.

2. On Siegel sets, we consider coefficients of differential forms with respect to special frames, as in [2]. For $\lambda \in X(S)$ we say that $\eta \in \Omega_{\lambda+}(\Gamma\backslash X)$ if the coefficients of η and of $d\eta$ satisfy a growth condition,

$$|f(x)| < a(x)^{\lambda}\,|P(\log a^{\alpha_1}, \ldots, \log a^{\alpha_l})|, \tag{1}$$

where $\alpha_1, \ldots, \alpha_l$ are the simple $\mathbf{Q}$-roots and P is a polynomial in l variables ($l = \dim S$). The proof of the following theorem is analogous to that of 7.4 in [2].

THEOREM 2. *If λ is dominant, then the injection $\Omega_{\lambda+}(\Gamma\backslash X) \longrightarrow \Omega(\Gamma\backslash X)$ induces an isomorphism in cohomology. The elements of $\Omega^q_{\lambda+}$ are square integrable if $q \leqslant c(G, \lambda)$. The space of square integrable harmonic q-forms contained in $\Omega_{\lambda+}(\Gamma\backslash X)$ maps injectively into the cohomology of Γ for $q \leqslant c(G, \lambda) + 1$. If $\lambda < 0$, then $H^*(\Omega_{\lambda+})$ is canonically isomorphic to the complex cohomology with compact supports of $\Gamma\backslash X$.*

3. Let M be a Riemannian manifold. Let $\Omega_{(2)}(M)$ be the complex of differential forms η on M such that η and $d\eta$ are square integrable. By definition $H_{(2)}(M) = H^*(\Omega_{(2)}(M))$ is the space of L^2-cohomology of M. (See [6], where equivalent Hilbert space definitions are given.) Let $\mathcal{H}_{(2)}(M)$ be the space of L^2-harmonic forms. It is known that if M is complete, then the natural map j: $\mathcal{H}_{(2)}(M) \longrightarrow H_{(2)}M$ is injective. If M is compact, then j is an isomorphism and $H_{(2)}(M) = H^*(M; \mathbf{C})$.

THEOREM 3. *There are canonical isomorphisms*

$$H_{(2)}(\Gamma\backslash X) = H^*(\mathfrak{g}, K; L^2(\Gamma\backslash G)^{\infty})$$

and

$$H_{(2)}(\Gamma\backslash G) = H^*(\mathfrak{g}; L^2(\Gamma\backslash G)^{\infty}).$$

As usual, if (π, V) denotes a unitary representation of $G(\mathbf{R})$ then V^{∞} denotes the space of C^{∞}-vectors in V. To establish Theorem 3, one proves first the second statement using a homotopy operator defined by the convolution by a compactly supported smooth function on G, and then deduces the first one by the comparison theorem for spectral sequences, applied to suitable spectral sequences in relative Lie algebra cohomology.

4. The space $L^2(\Gamma\backslash G)$ is the sum of the discrete spectrum $L^2(\Gamma\backslash G)_d$ and the continuous spectrum $L^2(\Gamma\backslash G)_{ct}$. By results obtained jointly with H. Garland [3], [4], $\mathcal{H}_{(2)}(\Gamma\backslash X)$ is finite dimensional and is the direct sum of the spaces $H^*(\mathfrak{g}, K; H_i^\infty)$, where H_i runs through a set of irreducible constituents of $L^2(\Gamma\backslash G)_d$. By [8], $L^2(\Gamma\backslash G)_{ct}$ is a Hilbert direct sum of invariant subspaces, say V_i ($i \in I$), each of which is a continuous integral of unitarily induced principal series (from parabolic **Q**-subgroups). By [4], $H^*(\mathfrak{g}, K; L^2(\Gamma\backslash G)_{ct}^\infty)$ is the sum of the $H^*(\mathfrak{g}, K; V_i^\infty)$ and can be nonzero only for finitely many terms. Those spaces can be computed as in [5, III] and can be nonzero only if the underlying parabolic subgroup is fundamental [5, IV] in $G(\mathbf{R})$. Together with Theorem 3, this proves

THEOREM 4. *The map* j: $\mathcal{H}_{(2)}(\Gamma\backslash X) \longrightarrow H_{(2)}(\Gamma\backslash X)$ *is an isomorphism if* G *has no proper parabolic* **Q**-*subgroup which is fundamental in* $G(\mathbf{R})$, *in particular if* rank G = rank K.

5. It is rather likely that if G has a proper fundamental parabolic subgroup P_1 defined over **Q**, then Γ has a subgroup Γ' of finite index such that $H_{(2)}(\Gamma'\backslash X)$ is infinite dimensional. This has been checked in a number of cases: (i) $G = \mathbf{SO}(n, 1)$ for $n \geqslant 3$ odd (with $\Gamma = \Gamma'$); (ii) the group P_1 is minimal over **R**; (iii) $G = \mathbf{SL}_n(\mathbf{R})$ and $\Gamma \subset \mathbf{SL}_n(\mathbf{Z})$. In those cases, infinite-dimensional cohomology occurs exactly in the dimensions q such that $\dim X - l_0 < 2q \leqslant \dim X + l_0$, where $l_0 = \operatorname{rank} G - \operatorname{rank} K$.

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