BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, July 1980

HOMOGENEOUS EXTENSIONS OF C^*-ALGEBRAS AND K-THEORY. I[1]

BY CLAUDE SCHOCHET

Let L denote the bounded operators on a complex, separable, infinite-dimensional Hilbert space, K the ideal of compact operators, $Q = L/K$ the Calkin algebra, and π: $L \longrightarrow Q$ the natural map. Brown, Douglas, and Fillmore (BDF) [1], [2] initiated the study of unitary equivalence classes of extensions of C^*-algebras of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

for fixed separable nuclear C^*-algebras A. The resulting group of equivalence classes is denoted $\operatorname{Ext}(A)$, or $\operatorname{Ext}(X)$ when $A = C(X)$, the ring of continuous complex-valued functions on a compact metric space X. In [2], BDF show that $\operatorname{Ext}(X) \cong K_1(X)$ when X is a finite complex. If X is of finite dimension then $\operatorname{Ext}(X)$ has been calculated by Kahn, Kaminker, and the author (KKS) [3]:

$$\operatorname{Ext}(X) \cong {}^sK_1(X) \overset{\text{def}}{\equiv} K^0(FX)$$

where ${}^sK_*(X) = K^*(FX)$ is Steenrod K-homology and FX is a CW-approximation for the function spectrum $\{F(X, S^n)\}$. In particular, if X is a closed subset of S^{2n} then

$$\operatorname{Ext}(X) \cong [S^{2n} - X, Q^r] \equiv K^0(S^{2n} - X)$$

where Q^r denotes the group of invertible elements of Q with the subspace topology, and $[X, Y]$ denotes basepoint-preserving homotopy classes of based maps $X \longrightarrow Y$. *Henceforth X and Y are understood to be finite-dimensional compact metric spaces.*

For a topological space Y and $*$-algebra B, the continuous functions $C(Y, B)$ form a $*$-algebra. In particular, we consider the algebra $C(Y, L_{*s})$, where L_{*s} denotes L with the strong-$*$ topology. This is a C^*-algebra with

---

Received by the editors November 7, 1979 and, in revised form, December 6, 1979.

1980 *Mathematics Subject Classification.* Primary 46L05, 55N15; Secondary 46M20, 47C15, 55N07, 55N20, 55P25, 55U25.

Key words and phrases. Extensions of C^*-algebras, Brown-Douglas-Fillmore theory, Steenrod homology, K-homology theory.

[1]Research partially supported by the National Science Foundation.

0002-9904/80/0000-0304/$02.00

C^*-norm equal to the sup norm provided that Y is compact metric. Let $\mathcal{Q}(Y) = C(Y, L_{*s})/C(Y, K)$ be the quotient C^*-algebra.

Pimsner, Popa, and Voiculescu (PPV) [6], [7] consider a substantial generalization of the BDF work. They consider unitary equivalence classes of extensions of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(Y, K) & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \tau & & \\ 0 & \longrightarrow & C(Y, K) & \longrightarrow & C(Y, L_{*_s}) & \longrightarrow & \mathcal{Q}(Y) & \longrightarrow & 0 \end{array}$$

which are homogeneous; the various restrictions

$$A \xrightarrow{\tau} \mathcal{Q}(Y) \xrightarrow{p_y} \mathcal{Q}$$

are extensions. If A is a separable nuclear C^*-algebra then a group $\mathrm{Ext}(Y; A)$ is obtained, and if Y is a based space with basepoint y_0 then a reduced group $\mathrm{Ext}(Y, y_0; A)$ is obtained.

In this note we determine the groups $\mathrm{Ext}(Y, y_0; C(X))$, denoted $L(Y, X)$ for brevity.

Define $L_n(Y, X)$ by

$$L_n(Y, X) = \begin{cases} L(Y, X) & n \text{ odd}, \\ L(Y, SX) & n \text{ even}. \end{cases}$$

Results of PPV imply that for X fixed, $L_*(-, X)$ is a cohomology theory and that for Y fixed, $L_*(Y, -)$ is a Steenrod homology theory [4]. Furthermore, Bott periodicity is satisfied in each variable, and there is an interchange isomorphism $L_*(Y, SX) \cong L_*(SY, X)$.

THEOREM 1. *Let X and Y be finite-dimensional compact metric spaces. Then there exist isomorphisms*

$$\Gamma_*^{Y,X}: L_*(Y, X) \longrightarrow K^*(Y \wedge FX)$$

such that

(1) *for each fixed Y, Γ_* is a natural equivalence of Steenrod homology theories,*

(2) *for each fixed X, Γ_* is a natural equivalence of cohomology theories,*

(3) *when $Y = S^0$ the natural equivalence corresponds to the* KKS *natural equivalence $\mathrm{Ext}_*(X) \cong K^*(FX)$,*

(4) *the natural diagram of isomorphisms*

$$\begin{array}{ccc} L_*(Y, S^2X) & \longrightarrow & K^*(Y \wedge F(S^2X)) \\ \downarrow & & \downarrow \\ L_*(S^2Y, X) & \longrightarrow & K^*(S^2Y \wedge FX) \end{array}$$

commutes, where the vertical maps are the interchange isomorphisms and the horizontal maps are instances of Γ_*. *(The* K^* *interchange is induced by the natural equivalence* $Y \wedge F(S^2X) \cong S^2Y \wedge FX$.)

Theorem 1 is the principal result; the other theorems are more or less immediate consequences of it. The proof of Theorem 1 follows the pattern of proof of [3, Theorem C].

THEOREM 2. *If X is a closed subset of S^{2n} then there is an isomorphism*

$$L(Y, X) \cong K^0(Y \wedge (S^{2n} - X)).$$

This is obvious given the fact that the suspension spectrum of $(S^{2n} - X)$ will serve stably as a replacement for FX. To completely destabilize, use the identification of $K^0(W)$ with $[W, \underset{\sim}{Q}^r]$ and obtain

$$L(Y, X) \cong [Y \wedge (S^{2n} - X), \underset{\sim}{Q}^r].$$

For computations when Y is a finite complex the following theorem is particularly useful.

THEOREM 3 (KÜNNETH THEOREM). *Suppose that $K^*(Y)$ is finitely generated as a graded abelian group. Then there is a natural short exact sequence*

$$0 \longrightarrow (K^1(Y) \otimes {}^sK_0(X)) \oplus (K^0(Y) \otimes {}^sK_1(X)) \xrightarrow{\alpha} L(Y, X) \longrightarrow$$

$$\xrightarrow{\beta} \mathrm{Tor}(K^1(Y), {}^sK_1(X)) \oplus \mathrm{Tor}(K^0(Y), {}^sK_0(X)) \longrightarrow 0$$

In particular, if $K^*(Y)$ is free abelian (e.g., if $Y = S^n$, CP^n, $G_k(C^n)$, $U(n)$, G a compact connected, simply-connected Lie group) then α is an isomorphism. If X is a finite complex then the sequence splits unnaturally.

Theorem 3 allows us to express $L(Y, X)$ in terms of $K^*(Y)$ and ${}^sK_*(X) \cong \mathrm{Ext}_*(X)$. For example, if X and Y are subsets of the plane and Y is a finite complex with $(n + 1)$ path components, then

$$L(Y, X) = (Z^n \otimes [C - X, Z]) \oplus ([Y, S^1] \otimes \hom([X, Z], Z)).$$

Note that $L(S^1, S^0) \cong 0 \oplus Z \cong Z$. This group stores the obstruction to the following lifting problem. If $f: S^1 \longrightarrow \underset{\sim}{Q}$ is a continuous function which takes values in the projections of $\underset{\sim}{Q}$ then each $f(\lambda)$ may be lifted to a projection in $\underset{\sim}{L}$. (This corresponds to the first summand in $L(S^1, S^0)$ vanishing.) However there is an obstruction to finding a *continuous* projection-valued lift for f measured by the second summand. This obstruction is not immediate from the BDF analysis.

G. G. Kasparov has also studied extensions of C^*-algebras by somewhat different techniques. He has announced [5] an isomorphism involving his theory K_*K:

$$K_*K(C(X), C(Y)) \cong K^*(Y \wedge DX)$$

for finite complexes X and Y, where DX is the classical Spanier-Whitehead dual. Theorem 1 then implies that

$$L_*(Y, X) \cong K_*K(C(X), C(Y))$$

for finite complexes. Thus the Kasparov and PPV machines do coincide on finite complexes.[2]

The author wishes to express his gratitude to R. G. Douglas, E. Effros, and J. Kaminker for continuing support, and his special gratitude to D. Voiculescu, for sharing with him preliminary versions of his work and for initially telling him about it on the steps of Porthania.

Details of this work will, of course, appear elsewhere.

REFERENCES

1. L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 58–128.

2. ———, *Extensions of C*-algebras and K-homology*, Ann. of Math. (2) **105** (1977), 265–324.

3. D. S. Kahn, J. Kaminker and C. Schochet, *Generalized homology theories on compact metric spaces*, Michigan Math. J. **24** (1977), 203–224.

4. J. Kaminker and C. Schochet, *K-theory and Steenrod homology: applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc. **227** (1977), 63–107.

5. G. G. Kasparov, *K-functor in the extension theory of C*-algebras*, Funkcional. Anal. i Priložen. **13** (1979), 73–74.

6. M. Pimsner, S. Popa and D. Voiculescu, *Homogeneous C*-extensions of C(X) ⊗ K(H)*, Part I, J. Operator Theory **1** (1979), 55–108.

7. ———, *Homogeneous C*-extensions of C(X) ⊗ K(H)*, Part II, J. Operator Theory (to appear).

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

Temporary address for academic year 1979–1980: Department of Mathematics, University of California at Los Angeles, Los Angeles, California 90024

[2]Jonathan Rosenberg and the author have shown recently that if Y is a locally compact subset of R^n and A is a separable nuclear (not necessarily unital) C^*-algebra, then the Kasparov group $\mathcal{E}xt(A, C(Y))$ which classifies *all* extensions of the form $0 \to C(Y, K) \to E \to A \to 0$ is isomorphic to the PPV group $\mathcal{E}xt(Y^+, +; A^+)$, where Y^+ is the one-point compactification of Y and A^+ is the unitalization of A.