technical proofs of embedding theorems might have been referenced rather than proved.

Each chapter ends with bibliographical comments. As some sections of the book closely follow the original papers these comments should have pointed out exactly who proved what, but often fail to do so.

There are the usual wealth of misprints and a few errors. For example, on page 3 a result of Browder is "proved". However, as soon as they deviate from Browder's correct proof they say "Since a normed linear space is separable if and only if its dual is, (Dunford and Schwartz p. 65) . . . ", which is false. The cited reference states the correct version. The proof on page 16 contains a slip (misprint?) and on page 281 it is stated that the truncation of a function in the Sobolev space $W^{m,p}$ also lies in the space: this holds only if $m = 1$ (or 0).

The book contains much material previously unavailable in book form. Some of the subjects are far from closed and developments have occurred since the book's publication. The book can well be read by someone who wishes to "get into" this subject. Whether it can be used in university courses, as the authors hope, is less clear.

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Differentiation of real functions, by Andrew M. Bruckner, Lecture Notes in Math., vol. 659, Springer-Verlag, Berlin and New York, 1978, x + 246 pp., $12.00.

For most of us, the extent of our knowledge of the differentiation theory of real functions is quite limited. The standard information may be classified as follows:

(i) Derivatives share some of the properties of continuous functions, e.g., they have the intermediate value (Darboux) property.

(ii) The primitive can be reconstructed from a derivative if the derivative is summable.

(iii) Monotone functions are differentiable a.e., but continuous functions are typically (except for a subclass of first category) nowhere differentiable.

This is not to belittle the state of our knowledge. Items (ii) and (iii) subsume many topics, e.g., absolutely continuous and singular functions, the Lebesgue decomposition theorem, and many facts about Dini derivatives and right and left derivatives. However, this constitutes a small portion of the state of the theory as presented in the *Theory of the integral* by Saks [**13**] more than forty years ago, and this theory has expanded greatly in the interim, particularly after the seminal paper of Zahorski in 1950 [**18**].

Zahorski hoped to characterize classes of derivatives by conditions on their associated sets, i.e., sets of the form $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$. He succeeded in characterizing the associated sets of certain classes of derivatives, and the wealth of new ideas and new perspectives he brought to these problems greatly stimulated work in differentiation theory. It is fortunate, therefore, that he did not observe at the onset of his investigations that the problem posed was not solvable. If h is a homeomorphism of R onto R, say h increasing, then $\{x: h \circ f(x) < a\} = \{x: f(x) < h^{-1}(a)\}$. Thus if a class of functions is to be characterized in terms of associated sets, the class must be closed under outside composition with homeomorphisms. It is not difficult to see that the various classes of derivatives are not closed under such compositions.

The derivatives (Δ) are in Baire class one $(\mathcal{B}_1)$ and, therefore, their associated sets are of type F_σ. Zahorski defined hierarchies of six classes of F_σ sets, M_k, $k = 0, 1, \ldots, 5$, and the six corresponding classes of functions $\mathfrak{M}_k$ given by $f \in \mathfrak{M}_k$ if its associated sets are in M_k. The sets M_k are defined in terms of the "density" of a set $E \in M_k$ at each of its points. For example, $E \in M_0$ if every $x \in E$ is a point of bilateral accumulation of E, and replacing "accumulation" by "condensation" yields the definition of M_1. A set E is in M_5 if it is an open set in the density topology, i.e., $x \in E$ implies $\lim_{|I| \to 0} |I \cap E|/|I| \to 1$ where I is an interval containing x. This is the same as saying E has metric density one at each of its points. The approximately continuous functions $(\mathcal{C})$ are then seen to be those which are continuous in the density topology. If $\mathfrak{D}$ denotes the class of functions with the Darboux property, it is easily seen that

$$\mathfrak{D}\mathcal{B}_1 = \mathfrak{M}_0 = \mathfrak{M}_1 \supsetneqq \mathfrak{M}_2 \supsetneqq \mathfrak{M}_3 \supsetneqq \mathfrak{M}_4 \supsetneqq \mathfrak{M}_5 = \mathcal{C}.$$

It can be shown that $\Delta \subsetneqq \mathfrak{M}_3$ and, letting b denote the class of bounded functions, $b\Delta \subsetneqq \mathfrak{M}_4$. Actually, $\mathfrak{M}_4$ *is* the class of associated sets of the bounded derivatives, but this does *not* say that f is a bounded derivative if its associated sets are in $\mathfrak{M}_4$. Preiss [**12**] has recently characterized the associated sets of the finite derivatives. Lipiński [**6**] had shown that they are a proper subset of $\mathfrak{M}_3$.

The problem of characterizing derivatives, i.e., of finding a condition on f which is necessary and sufficient for f to belong to a certain class of derivatives, has not been "satisfactorily" solved. It is useful, in considering this problem, to think of how the Lebesgue integral is characterized. The

Lebesgue integral is characterized by the property of absolute continuity. Another type of characterization is given by the Banach-Zarecki theorem **[8]**: F is an integral if and only if F is continuous, of bounded variation, and maps zero sets into zero sets (Lusin's condition N). This last result is interesting in that it shows exactly how a continuous function of bounded variation can fail to be absolutely continuous.

Neugebauer **[9]** has obtained a beautiful theorem which not only characterizes Δ but also $\mathfrak{D}\mathfrak{B}_1$ and shows exactly how a function in $\mathfrak{D}\mathfrak{B}_1$ can fail to be in Δ. It is "unsatisfactory" only in that the conditions involve the introduction of an auxiliary function, a function of intervals. If f is defined on a closed interval I_0, then $f \in \mathfrak{D}\mathfrak{B}_1$ on I_0 if and only if every nondegenerate closed interval $I \subset I_0$ contains an interior point x_I such that for every $x \in I_0$, $x \in I$ and $|I| \to 0$ implies $f(x_I) \to f(x)$. The function $f \in \Delta$ on I_0 if and only if $f \in \mathfrak{D}\mathfrak{B}_1$ and, whenever I, J, H are subintervals of I_0 for which $I = J \cup H$ and J and H are nonoverlapping, then, with x_I defined as in the condition for $\mathfrak{D}\mathfrak{B}_1$, we have

$$f(x_I) = (f(x_J)|J| + f(x_H)|H|)/|I|.$$

Much detailed study of the behavior of derivatives has been done. One way to do this is in the context of examining closely related classes of functions. We have observed that $\Delta \subset \mathfrak{D}$. There are, however, striking differences between these classes. While a derivative need not be continuous and need not assume extrema on compact sets, it does map connected sets onto connected sets and has a connected graph. Further, Δ is closed under addition and uniform limits, but not under multiplication or composition. A Darboux function need not have a connected graph. The sum of a continuous function and a Darboux function need not be in $\mathfrak{D}$ and, even more remarkably, *every* function is the sum of two functions in $\mathfrak{D}$.

We have observed that $\Delta \subset \mathfrak{B}_1$ and so it is natural to consider the class $\mathfrak{D}\mathfrak{B}_1$. For a function $f \in \mathfrak{B}_1, f \in \mathfrak{D}$ if and only if f has a connected graph. Note that a function in $\mathfrak{D}\mathfrak{B}_2$ need *not* have a connected graph. Functions in $\mathfrak{D}\mathfrak{B}_1$ combine better than do functions in $\mathfrak{D}$ and $\mathfrak{D}\mathfrak{B}_1$ is closed under uniform limits.

There are the following additional relations between the classes we have discussed:

$$b\mathfrak{A} \subsetneqq b\Delta, \quad \mathfrak{A} \subset \mathfrak{D}\mathfrak{B}_1, \quad \mathfrak{A} \not\subset \Delta.$$

It is interesting that if lower (or upper) semicontinuity is hypothesized, then $b\mathfrak{A} = b\Delta$.

Although a derivative may be discontinuous a.e., it must be continuous on a dense set of type G_δ. In fact, *every* G_δ-set which is dense on an interval is the set of points of continuity of a derivative. Nevertheless, a typical bounded derivative is aproximately discontinuous on a dense set (here $b\Delta$ is topologized by the sup norm).

One class of derivatives which has been characterized is that of a.e. derivatives. If f is the derivative of a function a.e., then f is measurable and a.e. finite. Conversely, if f is measurable and a.e. finite, then there exists a continuous F such that $F' = f$ a.e. For $f \in L^1$, this follows from the Fundamental Theorem of Calculus; for Denjoy integrable f it follows in the same manner. That it is true without additional hypotheses is a theorem of Lusin [13].

We have already made mention of the effect on differentiability of composition with homeomorphisms. This problem has been much studied recently. A function F is said to satisfy condition S' if for every interval J in the range of F we have

$$0 < \inf\{|E|: \text{measurable } E \text{ with } F(E) \supset J\}.$$

For continuous F, Fleissner and Foran [3] have shown that F satisfies S' is equivalent to each of the following:

(i) There is an h such that $(h \circ F)' \in b\Delta$.

(ii) There is an h such that $(h \circ F)' \in \Delta$.

Here h denotes a homeomorphism of R onto itself.

Composition on the inside with homeomorphisms (change of variable) has also proven interesting. Bruckner and Goffman [2] have shown that, for F defined on $[0, 1]$, the necessary and sufficient condition for there to exist a homeomorphism of $[0, 1]$ onto itself such that $(F \circ h)' \in b\Delta$ is that F be continuous and of bounded variation. Laczkovich and Petruska [5] have characterized the h such that $f \circ h \in \Delta$ for each $f \in \Delta$.

Many notions of generalized derivative have been defined. One notion arises from the question: Given a function f and a real sequence $h_n \to 0$, $h_n \neq 0$, when does there exist an F such that $(F(x + h_n) - F(x))/h_n \to f(x)$? For any f, there is a continuous F such that this holds with $\{h_n\}$ depending on x. This is a consequence of a theorem of Jarnik [4]. If it holds with a continuous F and $\{h_n\}$ depending only on f, then $f \in \mathfrak{B}_1$. It is not known if for every $f \in \mathfrak{B}_1$ there is a continuous F and a $\{h_n\}$ such that this holds. Some remarkable results concerning this notion of differentiability are known however.

Sierpiński [14] has shown that for every f finite on R and every $\{h_n\}$ there is an F such that $(F(x + h_n) - F(x))/h_n \to f(x)$ for every x. A later result of Eilenberg and Saks shows that finiteness is not necessary. Marcinkiewicz [7] has shown that there exist *universal generalized antiderivatives*, i.e., given $\{h_n\}$, there is a function F such that, for any a.e. finite and measurable f, there is a subsequence $\{h_{n_k}\}$ such that $(F(x + h_{n_k}) - F(x))/h_{n_k} \to f(x)$ a.e. In fact, a typical continuous function is such a function F.

The approximate derivative is probably the most extensively studied generalized derivative [13, Chapter 7]. We say that $\lim \operatorname{ap}_{x \to x_0} F(x) = c$ if there is a measurable set E of density one at x_0 such that $\lim_{x \to x_0, x \in E} F(x) = c$. Then the approximate derivative of F at x_0 is defined by

$$F'_{\mathrm{ap}}(x_0) = \lim_{x \to x_0} \operatorname{ap}(F(x) - F(x_0))/(x - x_0).$$

Assuming that the approximate derivative is finite, the class of approximate derivatives is in $\mathfrak{D}\mathfrak{B}_1$, in fact, it is in $\mathfrak{M}_3$, and it contains the finite derivatives properly; an approximate derivative is an ordinary derivative on a dense open set and will be a derivative if it is bounded either above or below by a derivative. From the Darboux property we see that if $F'_{\mathrm{ap}}(x)$ exists and is finite for all x on an interval I, then to $a, b \in I$, there corresponds a c

between a and b such that $(F(b) - F(a))/(b - a) = F'_{ap}(c)$, the Mean Value Theorem for approximate derivatives. Weil and O'Malley **[10]**, **[11]**, **[15]**, **[16]** have studied $\{x\colon \alpha < F'_{ap}(x) < \beta\}$ and obtained deeper results on the oscillatory behavior of the approximate derivative. Whitney **[17]** has shown that F is approximately differentiable a.e. on I if and only if for every $\epsilon > 0$ there is a closed set $E \subset I$ and a $\mathcal{C}^1$ function G such that $|I - E| < \epsilon$ and $F = G$ on E.

In our remarks above we noted that the typical continuous function is a universal generalized antiderivative. This result and many others concerning such topics as monotonicity and nondifferentiability lead one to the study of level sets of typical continuous functions. Bruckner and Garg **[1]** have gone beyond this question to produce an elegant determination of the exact manner in which a typical continuous function intersects nonvertical straight lines. If a countable set of exceptional directions is omitted, then the level sets in any direction are nowhere dense perfect sets except for the extreme level sets, which are singletons, and the level sets on a countable dense collection of levels, which are each the union of a nowhere dense perfect set and an isolated point. The level sets in the exceptional directions are also completely characterized.

The book under review provides a lively exposition of all the topics mentioned above and many more. The first third of the book is largely devoted to a careful discussion of basic properties of derivatives and Dini derivates. It is not only a review of classical results; modern material is presented and the new is well integrated with the old. Next the reconstruction of the primitive is discussed. The main result here is that for a finite derivative on an interval, a primitive can be constructed in an at most countable number of steps. The method relies on the ideas of the Denjoy-Khintchine construction of an integral. The Zahorski classes and the problem of characterizing derivatives are discussed in the next two chapters. Almost everywhere derivatives and other generalizations follow. The remaining chapters deal with transformations via homeomorphisms, generalized derivatives, monotonicity, stationary and determining sets, behavior of typical continuous functions, and miscellaneous topics.

This book presents, in a masterful way, a complete but concise development of the present state of knowledge of the subject and, as such, is invaluable to all who work in this field. But this book is invaluable in another respect. It not only offers a wealth of new problems, but the author shares so well with the reader the patterns of thought which give rise to these problems, that even a novice should be able to formulate meaningful conjectures of his own. This methodology is made particularly clear in the chapters on monotonicity and transformations via homeomorphisms.

We recommend this book strongly to anyone with an interest in real analysis. An expert in this area will surely find much that is of value, while the novice will find clear paths for the exploration of new and exciting vistas.

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Elements of homotopy theory, by George W. Whitehead, Graduate Texts in Math., vol. 61, Springer-Verlag, New York and Berlin, 1978, xxii + 744 pp.

Good news! George Whitehead has completed Volume 1 of the great American encyclopaedic treatise on homotopy-theory.

The approach adopted in this book is well described by the author.

"As the title suggests, this book is concerned with the elementary portion of the subject of homotopy theory. It is assumed that the reader is familiar with the fundamental group and with singular homology theory ....

"Anyone who has taught a course in algebraic topology is familar with the fact that a formidable amount of technical machinery must be introduced and mastered before the simplest applications can be made. This phenomenon is also observable in the more advanced parts of the subject. I have attempted to short-circuit it by making maximal use of elementary methods. This approach entails a leisurely exposition in which brevity and perhaps elegance are sacrificed in favour of concreteness and ease of application ....

"It is a consequence of this approach that the order is to a certain extent historical ....

"As I have stated, this book has been a mere introduction to the subject of homotopy theory. The rapid development of the subject in recent years has been made possible by more powerful and sophisticated algebraic techniques. I plan to devote a second volume to these developments."