the major theorem of the paper was not the proof of the denumerability of the set of algebraic numbers, but the nondenumerability of the reals.

Earlier historians, E. T. Bell in particular, have claimed that antisemitism was at the root of much of the opposition to Cantor's work. But as Dauben clearly establishes, Georg Cantor was not of Jewish ancestry; he was baptized a Lutheran and remained a devout Christian throughout his life. We obtain a deeper understanding of the nature of modern mathematics if we look to the mathematician *qua* mathematician for the source of the opposition.

It is consistent with the known facts that Kronecker's unwavering opposition to Cantor's work was the result of a total and fundamental difference of opinion as to the nature of mathematics. The extent of this difference can be seen in two aphorisms: Kronecker's "God made the integers; all else is the work of man" and Cantor's "The essence of mathematics is its freedom". For Kronecker the objects of mathematical investigation were the integers; these were fixed and unchanging. The mathematician's role was limited to the investigation of constructions built upon these eternal god-created forms. Creativity of new forms was not part of the province of the mathematician.

Cantor saw things differently. He knew that he could understand only if he had the freedom to create the forms and concepts which would encapsulate what he sought to understand. Dauben recognizes this, writing that the most important feature of his mathematical ability was "the capacity for creating new forms and concepts when existing approaches failed".

If we are to fully understand Cantor's influence on the nature of mathematical activity it is necessary to see Kronecker as belonging to the mathematical mainstream. It may be true that in his insistence that only the integers possessed an independent existence, he cast his net too narrowly, but the prevailing mathematical opinion then, as it had been since before Plato, was that the essence of mathematical activity is investigative, not creative. Philosophers still hold to this view, being far more concerned with epistemological matters than with ontological ones. However, after initial opposition, mathematicians were quick to appreciate the freedom that Cantor's conception of mathematics offered; as Hilbert wrote in 1925: "No one shall expel us from the paradise which Cantor created for us".

Just as Prometheus stole fire from the gods and instructed the human race in its use, so Cantor showed us that, like Kronecker's God, we too are free to create symbolic forms. The integers may be theogenic; since Cantor the rest of mathematics has become anthropogenic.

STANLEY H. STAHL

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Integral representations of functions and imbedding theorems, by Oleg V. Besov, Valentin P. Il'in, and Sergei M. Nikol'skiĭ, with an introduction by Mitchell H. Taibleson, V. H. Winston & Sons, Washington, D. C., vol. I, 1978, viii + 245 pp., vol. II, 1979, viii + 311 pp., $19.95 per volume.

This book (hereinafter referred to as *Integral representations*) is closely related to, but (both in technique and content) independent of Nikol'skiĭ's [5] which Mitchell Taibleson reviewed in this Bulletin [8]. In [8], Taibleson gave a history of the subject, motivated some of the early questions, identified some of the major workers and theorems and dealt with a number of technical definitions and examples. We encourage the reader to refer to it. Here we shall take a less detailed perspective, but shall try to distinguish the work of Besov, Il'in and Nikol'skiĭ, and particularly the book *Integral representations*, from that which preceded it.

Let $f\colon \mathbf{R}^n \to \mathbf{C}$ be defined everywhere. Every calculus student learns (or at least is told) that separate continuity (resp. differentiability) of f does not imply its continuity (resp. differentiability) as a function of several variables. This failure takes place both qualitatively and quantitatively. The following entertaining example was noticed by Yudovich. Let

$$f(x_1, x_2) = x_1 x_2 \log\log\left(1/\left(x_1^2 + x_2^2\right)\right)$$

for x near $(0, 0) \in \mathbf{R}^2$. Then

(i) f is continuous,
(ii) f_{11} is continuous,
(iii) f_{22} is continuous

but

(iv) the weak derivative f_{12} is not even essentially bounded.

On the other hand, all counterexamples involve a lack of uniformity. For instance if $f\colon \mathbf{R}^n \to \mathbf{C}$ and there are constants A_k so that

$$\left\|(\partial/\partial x_i)^k f(x_1, \ldots, x_n)\right\|_{L^2(\mathbf{R}^n)} \leq A_k$$

for $i = 1, \ldots, n$ and $k = 0, 1, \ldots,$ then $f \in C^\infty$ (Proof: Plancherel's theorem implies that $\xi_i^k \hat{f}(\xi) \in L^2$ for all i, k, so $p(\xi)\hat{f}(\xi) \in L^2$ for all polynomials p so $p(\partial/\partial x)f \in L^2$ for all polynomials p, where differentiation is interpeted in the weak sense. Now the Sobolev imbedding theorem implies that $f \in C^\infty$ after correction on a set of null measures). The hypotheses may even be weakened to $(\partial/\partial x_i)^k f \in L^1_{\text{loc}}(\mathbf{R}^n)$ for all i, k.

As early as 1911, S. Bernšteĭn considered quantitative versions of this result. If $\alpha \notin \mathbf{Z}$ and f is Λ_α ($[\alpha]$ times continuously differentiable with all $[\alpha]$-order derivatives satisfying a Lipschitz condition of order $\alpha - [\alpha]$) uniformly in each variable separately, is $f \in \Lambda_\alpha(\mathbf{R}^n)$? Bernšteĭn proved that $f \in \Lambda_{\alpha-\varepsilon}(\mathbf{R}^n)$, any $\varepsilon > 0$. In 1948 Nikol'skiĭ removed the ε. Moreover, he obtained sharp estimates for mixed derivatives if one assigns a different degree of smoothness to each coordinate direction.

It is easy to see how problems of this kind can spawn an industry. For instance, if we assign to each coordinate direction x_i a Lebesgue space L^{p_i} and an index $k_i \in \mathbf{N}$, then we may consider those $f\colon \mathbf{R}^n \to \mathbf{C}$ which for each i have k_i weak derivatives with respect to x_i each of which lives in L^{p_i}. What is the greatest q so that such an f is in L^q? Is such a function in any Lipschitz space? Does such a function automatically satisfy a similar condition for larger p_i and smaller k_i? Which cross derivatives exist and in what L^p classes do they live?

These questions all originate in the basic work of Sobolev. Since we shall have future occasion to refer to this work, we give now an example of such a result.

THEOREM. *Let* $1 \leqslant p < q < \infty$. *Let* $f\colon \mathbf{R}^n \to \mathbf{C}$, $f \in L^p$, $\partial f/\partial x_j \in L^p(\mathbf{R}^n)$, $j = 1, \ldots, n$, *where the derivative is interpreted in the sense of distributions. Then* $f \in L^q(\mathbf{R}^n)$ *provided* $1/q = 1/p - 1/n$.

Properly formulated, this is a theorem about *continuous imbedding* of Banach spaces.

We now enumerate several other basic problems of function theory, all closely related to the imbedding problem, and all of which are treated in *Integral representations*. The authors discuss Lipschitz spaces, Nikol'skiĭ spaces, Besov spaces, Sobolev spaces, Campanato spaces and several others of their own invention. We shall use the symbols B or B_j to denote any of these (or any of the nonisotropic variants indicated above).

(1) THE COMPACTNESS PROBLEM. In case $B_1 \subseteq B_2$, is the inclusion map compact? (Examples: The Ascoli-Arzela Theorem, the Rellich Lemma.)

(2) THE TRACE PROBLEM. Consider the operator

$$r_{n,k}\colon f(x_1, \ldots, x_n) \to f(x_1, \ldots, x_k, 0, \ldots, 0)$$

assigning to functions in $C_c^\infty(\mathbf{R}^n)$ their restrictions in $C_c^\infty(\mathbf{R}^k)$. When can the domain be extended to some $B_1(\mathbf{R}^n)$ and correspondingly the range to some $B_2(\mathbf{R}^k)$ so that the extended operator is bounded? (First considered by Sobolev.)

(3) THE EXTENSION PROBLEM. If $G \subseteq \mathbf{R}^n$ is there a bounded linear operator

$$E\colon B(G) \to B(\mathbf{R}^n)$$

so that $E(f)|_G = f$? (This problem has been considered by Calderón, Stein and Hestenes. One sees that this is a question both of function theory and of geometry. Domains with cusps or whose boundaries bifurcate are bound to be problematic.)

(4) MULTIPLICATIVE INEQUALITIES (also called Landau inequalities). When does a logarithmically convex combination of norms majorize the norm on an intermediate space? An example is

$$\|f\|_{C^1(\mathbf{R})} \leqslant C\|f\|_{\sup}^{1/2} \cdot \|f\|_{C^2(\mathbf{R})}^{1/2}.$$

(This question has been considered by, among others, Landau, Gagliardo-Nirenberg, and Stein.)

All of these problems are basic to classical analysis, and find manifold applications in function theory and partial differential equations. For instance, one may interpret the regularity problem for the Laplacian as follows. Suppose that

$$\Delta u = \sum_i \partial^2 u/\partial x_i^2 = f \qquad (*)$$

and that we know that $f \in B$. What can we say about u? The equation $(*)$, roughly speaking, gives information about $\partial^2 u/\partial x_i^2$, $i = 1, \ldots, n$, from which we would hope to extract information about cross derivatives. So the question takes on the flavor of the original problem of Bernšteĭn. That this problem can be solved, i.e., that one can estimate all the derivatives of u (to some order) in terms of $\Delta u = f$, is the property of *coerciveness* of Δ. Perhaps the main application of the techniques of *Integral representations* which is included in the book is a number of theorems on coerciveness.

The Russian school has made major contributions to the study of the foregoing problems in the classical isotropic setting. Besov, Il'in and Nikol'skiĭ have played the central role, beginning with Nikol'skiĭ's 1948 paper, in the study of these problems in the category of the nonisotropic spaces mentioned at the beginning of this review.

The book [5] constitutes a report mainly on the work done in this subject for functions defined in all of $\mathbf{R}^n$. Indeed, it succeeded in resolving many of the natural questions of imbedding, compactness, and trace, provided examples to show that the results are sharp, and proved a number of approximation results. The book *Integral representations* is a logical successor to [5]. It addresses the above questions, and some additional ones, for functions defined on a subdoman $G \subseteq \mathbf{R}^n$. The new results are far more than formal transliteration of those in [5]: entirely new techniques are developed in *Integral representations*. Indeed, while [5] used the method of best approximation by entire functions of exponential type (or in the periodic case, by trigonometric polynomials), it is very unnatural to approximate functions on a domain G by entire functions.

It is reasonable, both historically and practically, to turn to integral representations when doing function theory on G. If a function can be represented as an integral of itself, there is much to be learned (consider the Cauchy and Poisson integral formulae). The idea is to force a quasi-explicit averaging kernel to carry the information.

Surely the most primitive integral representation formula is the fundamental theorem of calculus:

$$f(x) = f(0) + \int_0^x f'(t)\, dt.$$

As the authors state, all of their integral formulae are based on the following additional idea (which has roots going far back in the literature of harmonic analysis). Let

$$\varphi \in C_c^\infty(\mathbf{R}^n), \quad \int \varphi(x)\, dx = 1.$$

Define, for $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, and for $f \in L^p(\mathbf{R}^n)$ let $f_\varepsilon(x) = f * \varphi_\varepsilon(x)$. For $0 < \varepsilon < h$ write

$$f_\varepsilon(x) = f_h(x) - \int_\varepsilon^h \frac{d}{dt}(f_t(x))\, dt.$$

When $\varepsilon \to 0$, it is known (see [7]) that $f_\varepsilon \to f$ pointwise a.e., so that letting $\varepsilon \to 0$ formally on the right gives

$$f(x) = f_h(x) - \int_0^h \frac{d}{dt} f_t(x)\, dt.$$

The differentiation under the integral sign falls on the φ so that we have succeeded in expressing f as an integral (or average) of itself. By integrating by parts, or by adding and subtracting appropriate terms, we can express f as an integral of its derivatives, or as an integral of its finite differences (expressions like $f(x + h) - f(x)$). Sobolev already exploited these ideas to prove his imbedding and trace theorems. The contribution of *Integral representations* is to systematically represent any of

(i) f,
(ii) $(\partial/\partial x)^\alpha f$,
(iii) $f(x + h) - f(x)$, etc.

in terms of any of the others, and to do so in a fashion which allows the assignment of weights to each variable. We thereby have a handle for studying our basic function-theoretic problems in the category of isotropic *or* nonisotropic spaces.

Nonisotropic spaces arise naturally in studying, say, the boundary behavior for solutions to the heat equation, the $\bar{\partial}$-Neumann problem or the Dirichlet problem for the Laplace-Beltrami operator in the Bergman metric. The utility of imbedding theorems in proving regularity results, of compactness theorems in passing from *a priori* to more general estimates, of trace theorems in restricting from a domain to its boundary and so on, is well established. Therefore *Integral representations* should become an important reference. Its systematic treatment of the calculus of finite differences and integral formulae set it apart from other books in the field. Finally, since so much of the original research is in Russian, this English translation provides a valuable guide to the literature.

There is some difficulty with seeing the forest for the trees, since theorems are usually stated in maximum generality. The novice should be advised that a greater perspective may be gained by first reading Stein's *Singular integrals and differentiability properties of functions* [7] and also by having a look at the methods of [1], [2] and [5].

The reader who is not already convinced of the technical nature of this book should consider the following. Classically, function theory on domains $G \subseteq \mathbf{R}^n$ was studied under the auspices of a *cone condition*. That is, we assume that ∂G may be covered in a nice way by open sets $G_1, G_2, \ldots,$ and that to each G_j is assigned a cone Γ_j so that if $x \in \partial G \cap G_j$ then $\{x\} + \Gamma_j \subseteq G$. This is plausible because we wish to express functions on $G \cap G_j$ as averages over these cones. In the present work, cones are replaced by more general objects called *horns* (a typical horn is $\{(x_1, x_2) \in \mathbf{R}^2\colon 0 < \frac{1}{2}x_1^2 < x_2 < 2x_1^2\}$). At various points in *Integral representations*, the authors consider weak, strong, and regular ____ conditions (fill in the blank with "horn", "cone", "cube", or "rectangular parallelipiped"). So we have twelve kinds of domains, myriad function spaces, and n dimensions over which to sprinkle them. There are also five basic questions (and several less basic ones such as isomorphism of different isotropy classes) to consider. Thus the book, while it considers important material, is not to be approached casually.

The authors (and the translator) have done a fine job of giving us a readable presentation of this difficult material. In fact the book is essentially self-contained (if not well motivated). The work begins by reviewing a lot of basic function theory: fractional integration, singular integrals, the inequalities of Minkowski, Hölder, Hardy, etc. (Many of these are generalized to nonisotropic versions for the task at hand.) There is then a long chapter (76 pages) presenting all the integral representation formulae. A nice section on coercive estimates provides some applications of the theory. The remainder of the two volumes is organized, roughly speaking, according to the function spaces being studied rather than the function-theoretic questions being considered. From the point of view of readability, this seems to have been a good choice.

The style of the book is informal: the authors are not afraid to repeat definitions and formulas when they are needed (however the book would have benefited enormously from a list of notation and an extensive index). Most sections begin with a nontechnical description of what is about to be done. There are many enlightening comments describing alternate approaches to a given question (for instance the extension theorems are studied via integral formulae but the approaches of Hestenes and Stein are also mentioned) or simply explaining what is going on. Long calculations are usually broken up into plausible sequences of lemmas.

The numerous misprints, the bad translating, and the grotesque bibliographical transmogrification of [5] which Taibleson lamented in his review [8] have been avoided in *Integral representations*. (There are still a few uncomfortable habits of phraseology, probably unavoidable in any translation. The phrase "f is concentrated on G" is used instead of "f is supported on G". "A result of Hilbert", with no other identifying words, turns out to be the Nullstellensatz. The "Newton-Leibniz formula" is the authors' name for the fundamental theorem of calculus.) The vast improvement is no doubt due in large measure to Taibleson's efforts as translation editor of *Integral representations*.

It would have been an asset had the authors considered whether the various function spaces considered form real or complex scales of interpolation spaces. It would have been valuable to know whether the spaces are stable under singular integrals or fractional integrals of various kinds (some of this information is implicit in the proofs–it could have been made explicit). Numerous maximal functions and problems in the theory of differentiation of integrals lurk in the background of the discussions in the book; only the Jessen-Marcinkievicz-Zygmund theorem is mentioned. Interpolation theory could have been used to simplify and clarify some of the proofs.

We would also like to submit, not so much to criticize as to suggest further avenues of research, that the results of this book have a certain aura of artificiality. A class of functions whose definition is rigidly tied to the coordinate directions has little to do with a generic subdomain of $\mathbf{R}^n$. (The authors *do* consider function spaces on manifolds, but do *not* address this problem.) Even the unit disc satisfies only a very restricted family of "horn conditions", conditions which are necessary for most of the theorems in the book. What is perhaps more natural, and has wider applicability to partial differential equations, is to consider functions which satisfy different differentiability conditions along (possibly noncommuting) families of vector fields. These arise, for instance, in the $\bar{\partial}$-Neumann problem [3] and in the study of hypoelliptic operators [4], [6]. It would be valuable to have integral formulae for these classes of functions. We hope that future volumes might consider these matters.

Integral representations will prove a valuable reference for experts. By its very technical nature, it contains no results with the compelling elegance of, say, the Riemann mapping theorem. But the techniques are far more important than the specific results and, by that measure, the book is a success.

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STEVEN G. KRANTZ

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The algebra of random variables, by M. D. Springer, Wiley, New York, 1979, xxii + 470 pp., $26.95.

It is both surprising and regrettable that it took over 30 years after the appearance of the pioneering paper by B. Epstein in (1948), *Some applications of the Mellin transform in statistics*, to produce the first text on the *Algebra of random variables* which is based on an elaboration and extension of Epstein's ideas. It is also equally unfortunate and puzzling that Epstein's paper appeared so late in this century, over ten years after Cramér's classical *Random variables and probability distributions*. The book under review is indeed very close in its mathematical content to the treatises on the subject matter of the special functions which originated in the beginning of this century. In fact, the mathematics in this book could fit very well into Whittaker and Watson's *Modern analysis* (1915).

This lag of about half a century is–in my opinion–due to two basic reasons: the awkward and uncertain position of probability theory (and thus indirectly what is known today as "statistical distribution theory") within the framework of all the mathematical disciplines which lasted at least until the publication of Kolmogorov's axiomatization in 1933 and to some extent to a certain contempt exercised by the editors of some mathematical and statisti-