BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0014/$02.00

*Elliptic systems in the plane*, by W. L. Wendland, Pitman, London, San Francisco and Melbourne, 1979, xi + 404 pp., $57.50.

Because of the close relation of elliptic equations of two variables with the theory of analytic functions, with variational principles, and with some important aspects of physics and engineering, a vast amount of literature has been compiled in the last decade concerning the properties of solutions, representation theorems, boundary value theory, constructive and computational aspects. The list of authors of important papers on the theory of elliptic operators resembles a "who is who" list of mathematical analysis. Various aspects of the theory are identified with the names of L. Bers, Agmon, Douglis and Nirenberg, Hörmander, Lions and Magenes, N. Levinson, Vekua, Bitsadze, Lavrentiev, Sobolëv, F. Browder, Miranda, Fichera, Atiyah and Singer, Malgrange, H. Lewy.

In this monograph the author deliberately avoids the abstract theory of elliptic operators, and restricts himself to the study of the two-dimensional case. The problems studied are linear. A large section of the book is devoted to the study of the normal form

$$\left.\begin{aligned} u_x - v_y &= Au + Bv + c, \\ u_y + v_x &= \tilde{A}u + Bv + \tilde{c}, \end{aligned}\right\} \text{ in } G,$$

$$u \cos \tau(s) - v \sin \tau(s) = \phi \quad \text{on } \dot{G}, \tag{1}$$

where $G$ denotes a bounded, simply connected domain in $\mathbf{R}^2$ with a Hölder continuously differentiable, positively oriented boundary, $\dot{G} \in \mathcal{C}^{1+\alpha}$, $\alpha > 0$.

This system of equations becomes the Cauchy-Riemann system (C.R. equations) which may be concisely written $(\partial/\partial\bar{z})w = 0$ $(w = u + iv)$, if the right hand side is identically equal to zero. Because of this close relation with the C. R. equations boundary value problem of this type permit direct applications of techniques used in the theory of analytic functions, and the use of results concerning harmonic functions. The classical problem of Dirichlet and Neumann can be solved for the region $G$ if one knows Green's function, which is given in terms of boundary data only (on $\dot{G}$). Green's function can be found if the map $\phi(z)$ is known, where $\phi$ maps $G$ conformally into the unit circle. For example for the Dirichlet problem ($\Delta u = 0$ in $G$, $u$ is given on $\dot{G}$) the following formulas were known almost a century ago

$$G(z, \zeta) = \frac{1}{(2\pi)} \left| \frac{(\phi(z) - \phi(\zeta))}{(1 - \phi(z)\phi(\zeta))} \right|,$$

and

$$u(z) = \int_{\dot{G}} u(\zeta) \frac{\partial G(z, \zeta)}{\partial n}\, ds.$$

The last formula indicates that the Dirichlet problem can be replaced by an equivalent integral equation, if integration over the boundary of $G$ is replaced by an equivalent integral over the region $G$.

The codimension of the range of this integral operator $T$ is zero and the dimension of its null space is finite. Thus it is a Fredholm operator. The Fredholm "normality condition" applies. Namely, the equation $Tx = f$ has a solution if and only if $f$ is orthogonal to the null space of $T^*$, which is the adjoint of $T$. In general, equations of the form

$$u_x + Bu_y + Cu = f \tag{2}$$

which are elliptic (no eigenvalues of $B$ are real) can be transformed by means of the Green functions (or equivalently using the closely related concept of an elementary solution) to the Fredholm type integral equations. Generally the kernel of the integral operator is singular.

Let $p$ denote the eigenvectors of the transpose of $B$. In the $2 \times 2$ case (a two dimensional system of equations) exactly two eigenvectors exists $p_1$, $p_2$. Let $p\,\pm$ stand for

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \pm i \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \end{pmatrix}.$$

The following transformation has been known for some time, and turns up in many classical works (for example, I. N. Vekua gives a detailed account of its properties in *Generalized analytic functions*, Pergamon Press, Oxford, 1962) $w = U + iV = p_+^T u$.

The system (2) is transformed to an equivalent system of differential equations, while the boundary conditions are reduced to the form $\mathrm{Re}(2\mathbf{q} \cdot \mathbf{r})w = \chi = 0$ on $\dot{G}$, with $\mathbf{q}$ defined by $\mathbf{u} = 2\,\mathrm{Re}(w\mathbf{q})$. It is essential that $\mathbf{q} \cdot \mathbf{r} \neq 0$ on $\dot{G}$. This is the so-called Lopatinskiĭ condition, which turns out to be equivalent to "the complementing condition" of Agmon, Douglis and Nirenberg, and to "the covering condition" of Lions and Magenes. Various aspects of this classical theory were extended while dealing with systems of the normal form (1) in the earlier monograph of W. Haack and the author.

The presentation follows for a while closely the book of W. Haack and the author (*Lectures of Pfaffian and partial differential equations*, Pergamon Press, Oxford, 1972). Next, the author derives some generalizations of the Fredholm theory, using rather strongly the so-called Lopatinskiĭ condition. For second order systems having the Laplace operator as a principal part, the general theory presented here can be found in the monograph of Bitsadze (A. V. Bitsadze, *Boundary value problems* (translated from Russian), North-Holland, Amsterdam, 1968). However, the results presented here are more general than Bitsadze's. The properties of the fundamental solution for the operator $\partial/\partial\bar{z}_j$ lead to the study of corresponding singular integral equations. The author follows Vekua's ideas in the proofs of analytic continuation, but treating some cases which require some generalizations of Vekua's techniques.

The two currently most popular techniques of generating computational algorithms for solving problems described by partial differential equations are (1) The Rayleigh-Ritz-Galerkin techniques, (2) Finite element approximations. The common feature of Rayleigh-Ritz and Galerkin approach is the reduction of the function space (in which one approximates the solution) to a finite dimensional space. If the solution of a partial differential equation can be restated as a solution of a variational problem, the Galerkin approach generally reduces it to the finding of an approximate solution by solving a finite system of algebraic equations. An entirely different method attempts to approximate the model of a continuous physical system by a system containing only finitely many elements which obey simpler physical laws. The finite element method was originally introduced by engineers in the late 1930s in the study of shells and plates. Basically the continuous shell or plate was replaced by a simple structure which contained only pivoted joints. A remarkable connection was established in the 1950s between this purely engineering approach and the abstract theory of splines.

Efficient methods of approximating solutions of partial differential equations by finite elements can be found in the essay of P. G. Ciarlet and P. A. Raviart in *The mathematical foundations of finite element method with applications to partial differential equations*, K. Aziz, editor, pp. 409–474, Academic Press, New York, 1972, or in the article of J. A. Nitsche in the same symposium, pp. 603–627.

The second part of this monograph deals with numerical methods. The choice of topics follows closely the author's research interests in discussing the integral equation approach. The Galerkin-Ritz method is reviewed, and Fichera's treatment is followed to produce an asymptotic error analysis.

In the discussion of finite difference methods the author follows Bramble and Hubbard in the proof of a discrete maximum principle. A discrete analogue of Green's formula is established by referring to the results of Bramble, Hubbard, and Thomée. The rate of convergence result is due to Ciarlet. Finite element approximations and continuous approximations are briefly discussed in the last chapter. Some theorems (for example, the convergence of embedding Newton's method) are offered with a promise that the proofs will be published elsewhere. An appendix offers examples of application in which the author points out the applicability of the results discussed in the text, but stops short of discussing any details.

The editorial assessment on the inside cover states that this book is of interest to postgraduate research workers in pure and applied mathematics as well as to engineers and physicists. This assessment is somewhat optimistic. Unless the reader had some exposure to modern functional analysis, to functions of many complex variables, to singular integral equations, and to partial differential equations at a graduate level, he will find this monograph difficult to digest. For example, the reader is expected to know the definitions and properties of the Sobolëv spaces $H^s(G)$, $H^{r+1/2}(\dot{G})$, $H^{-1/2}(\dot{G})$, etc. For some reason the space $\mathring{H}^s(G)$ is defined in the very last chapter of the book. The trace theorem is used without stating it, but with a reference to the book of Lions and Magenes; the Bergman-Vekua operator formula is used without an explanation, and so on. This level of sophistication can hardly be expected from an engineer or a physicist. Clearly, this book is written for mathematicians.

Two basic criticisms may be offered of this book, regarded as a mathematical text. It tries to cover too much material, and it does not cover enough material. These statements are not incompatible. The chapter on finite difference methods could be either shortened, or omitted while preserving continuity of the material. In view of the vast amounts of literature which appeared in the last couple of years concerning the finite element approximation, the few pages devoted to this subject seem to be either superfluous, or completely inadequate even as an introduction to this topic. On the other hand, some topics seem to stop short of mentioning some important recent results. For example, the only recent results on continuation of solutions reported in this book seem to be restricted to the theorems of G. H. Hile and M. Protter.

It was a surprise to find that Gårding's inequality was never mentioned in a book on elliptic partial differential equations.

Some minor misprints exist. For example, Carleman's name is misspelled on p. 212. Also, the spelling of all Russian names leaves a lot to be desired. Olga Ladyženskaja is spelled as "Ladyshenkaja" in the index. Lavrentjev is spelled "Lavrentieff" on p. 40 and "Lavrentjev" in the index. Lopatinskiĭ is spelled in two different ways on consecutive lines in the index. The index seems to be a minor disaster area. Rellich's theorem is supposed to be on p. 379. It turns out to be on p. 369. The estimate of Astrahancev is supposed to be on p. 366, but it actually is quoted on p. 367. The reviewer selected 10 items at random. Four turned out to be misnumbered in the index. On the positive side, this book leaves a general impression of competence, it is well written, contains a lot of information, and a vast amount of bibliography. Each chapter contains references and a list of additional references, plus remarks concerning the additional references. For example, the chapter on elliptic boundary value problems contains 116 references, and the chapter on singular integral equations contains 119 references. The reviewer recommends this book as a supplement to courses in partial differential equations and as a useful addition to the library of any analyst.

VADIM KOMKOV

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0015/$03.00

*Introduction to modular forms*, by Serge Lang, A Series of Comprehensive Studies in Mathematics, No. 222, Springer-Verlag, Berlin and New York, 1976, ix + 261 pp.

Modular forms are generalizations of the trigonometric functions which have proved to be very useful in number theory, physics, and geometry. In their most obscene generality they go by the name of automorphic forms, and this name refers to functions on symmetric spaces, such as the Poincaré-Lobachevskii upper half plane $H$, which satisfy certain differential equations (usually those of Cauchy and Riemann) and exhibit invariance properties under a discrete group of isometries of $H$, such as the modular group $\mathrm{SL}(2, \mathbf{Z})$ of $2 \times 2$ matrices of determinant one and integer entries, acting on $H$ by fractional linear transformation. Number theorists, as will be seen, are often interested in congruence subgroups such as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}) \middle| c \equiv 0 \pmod{N} \right\}.$$