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# SUFFICIENCY OF McMULLEN'S CONDITIONS FOR $f$-VECTORS OF SIMPLICIAL POLYTOPES

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For convex $d$-polytope $P$ let $f_i(P)$ equal the number of faces of $P$ of dimension $i$, $0 \leqslant i \leqslant d-1$. $f(P) = (f_0(P), \ldots, f_{d-1}(P))$ is called the *f-vector of P*. An important combinatorial problem is the characterization of the class of all $f$-vectors of polytopes, and in particular of simplicial polytopes (i.e. those for which each facet is a simplex). McMullen in [5] conjectures a set of necessary and sufficient conditions for $(f_0, \ldots, f_{d-1})$ to be the $f$-vector of a simplicial $d$-polytope and proves this conjecture in the case of polytopes with few vertices. We sketch here a proof of the sufficiency[3] of these conditions, and derive in a related way a general solution to an upper bound problem posed by Klee.

The $f$-vectors of simplicial $d$-polytopes satisfy the *Dehn-Sommerville equations*

$$\sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i(P) = (-1)^{d-1} f_j(P), \qquad -1 \leqslant j \leqslant d-1,$$

where we put $f_{-1}(P) = 1$. As in [6, p. 170], for $d$-vector $f = (f_0, \ldots, f_{d-1})$ and integer $e \geqslant d$ let

$$g_j^{(e)}(f) = h_{j+1}^{(e)}(f) = \sum_{i=-1}^{j} (-1)^{j-i} \binom{e-i-1}{e-j-1} f_i, \qquad -1 \leqslant j \leqslant e-1,$$

with the convention that $f_{-1} = 1$ and $f_i = 0$ for $i < -1$ or $i > d-1$. We note here that these relations are invertible, allowing us to express the $f_i$ as nonnegative linear combinations of the $h_j^{(e)}(f)$. The Dehn-Sommerville equations for $f$ are, for any $e \geqslant d$, equivalent to $g_i^{(e)}(f) = (-1)^{e-d} g_{e-i-2}^{(e)}(f)$, $-1 \leqslant i \leqslant [e/2] - 1$. Let $h$ and $i$ be positive integers. Then $h$ can be written uniquely as

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[3]ADDED IN PROOF. R. Stanley has proved necessity since this was written.

$$h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Following McMullen put

$$h^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}$$

and define $0^{\langle i \rangle} = 0$. McMullen conjectured ([5], [6, p. 179]) that $(f_0, \ldots, f_{d-1})$ is the $f$-vector of a simplicial $d$-polytope if and only if the following three conditions hold:

$$g_i^{(d+1)}(f) = -g_{d-i-1}^{(d+1)}(f), \quad -1 \leq i \leq [\tfrac{1}{2}(d+1)] - 1, \tag{1}$$

$$g_i^{(d+1)}(f) \geq 0, \quad 0 \leq i \leq n-1, \tag{2}$$

$$g_i^{(d+1)}(f) \leq (g_{i-1}^{(d+1)}(f))^{\langle i \rangle}, \quad 1 \leq i \leq n-1, \tag{3}$$

where $n = [d/2]$. Condition (1) is just the set of Dehn-Sommerville equations, the conjectured necessity of (2) is known as the *Generalized Lower Bound Conjecture* ([7], [6, p. 178]).

We will sketch a proof of the following

THEOREM 1. *If $f = (f_0, \ldots, f_{d-1})$ satisfies* (1), (2), *and* (3) *above, then $f = f(P)$ for some simplicial d-polytope P.*

The case $d < 2$ is easily dispensed with, so assume $d \geq 2$. For finite $(d-1)$-dimensional simplicial complex $\Sigma$ let $|\Sigma|$ denote the underlying topological space of $\Sigma$. $f(\Sigma) = (f_0(\Sigma), \ldots, f_{d-1}(\Sigma))$ is the *f-vector of* $\Sigma$, where $f_i(\Sigma)$ is the number of $i$-dimensional simplices in $\Sigma$. For $e \geq d$ write $h^{(e)}(\Sigma)$ for $h^{(e)}(f(\Sigma))$. We call $h^{(d)}(\Sigma)$ the *h-vector of* $\Sigma$. This is equivalent to Stanley's $h$-vector of [9]. If $|\Sigma|$ is a $(d-1)$-sphere then the Dehn-Sommerville equations hold (see, for example, Grünbaum [1, p. 152]). If $|\Sigma|$ is a $d$-ball then $\partial|\Sigma|$ is a $(d-1)$-sphere with associated complex $\partial\Sigma$. $|\Delta|$ is then a $d$-sphere, where $\Delta = \Sigma \cup v \cdot \partial\Sigma$. It can be shown that $h_i^{(d+1)}(\Delta) = h_i^{(d+1)}(\Sigma) + h_{i-1}^{(d)}(\partial\Sigma)$, $0 \leq i \leq d+1$ (where we take $h_{-1}^{(d)}(\partial\Sigma) = 0$). The Dehn-Sommerville equations for $\Delta$ and $\partial\Sigma$ allow us to solve for $h_i^{(e)}(\partial\Sigma)$ in terms of $h_j^{(d+1)}(\Sigma)$. In particular [7]

$$h_i^{(d+1)}(\partial\Sigma) = h_i^{(d+1)}(\Sigma) - h_{d+1-i}^{(d+1)}(\Sigma), \quad 0 \leq i \leq [\tfrac{1}{2}(d+1)].$$

A nonvoid set $M$ of monomials $Y_1^{a_1} \cdots Y_s^{a_s}$ is said to be an *order ideal of monomials* if whenever $m_1 \in M$ and $m_2 | m_1$ then $m_2 \in M$. Let $\Phi$ be the set of all monomials in the variables $Y_1, \ldots, Y_s$. Give the elements of $\Phi$ the lexicographic linear order $<$ induced by $Y_1 < \cdots < Y_s$. A finite or infinite sequence $(H(0), H(1), \cdots)$ of nonnegative integers is said to be an *0-sequence* if there exists an order ideal $M$ of monomials in the variables $Y_1, \ldots, Y_s$ with each $\deg Y_i = 1$ such that $H(i) = \operatorname{card}\{m \in M: \deg m = i\}$. Stanley in **[10]** gives the following

THEOREM. *Let* $H: \mathbf{N} \longrightarrow \mathbf{N}$. *The following statements are equivalent*:

(i) $(H(0), H(1), \cdots)$ *is an* 0-*sequence.*

(ii) $H(0) = 1$ *and for all* $i \geqslant 1$, $H(i+1) \leqslant H(i)^{\langle i \rangle}$.

(iii) *Let* $s = H(1)$ *and for each* $i \geqslant 0$ *let* $M_i$ *be the first* (*in the ordering above*) $H(i)$ *monomials of degree* $i$ *in the variables* $Y_1, \ldots, Y_s$. *Define* $M = \bigcup_{i \geqslant 0} M_i$. *Then* $M$ *is an order ideal of monomials. Call* $M$ *the lexicographic order ideal of monomials associated with* $(H(0), H(1), \cdots)$.

IDEA OF PROOF OF THEOREM 1. If $(f_0, \ldots, f_{d-1})$ satisfies (1), (2), and (3), then by the above theorem $(H(0), \ldots, H(d+1))$ is an 0-sequence, where $H(i) = h_i^{(d+1)}(f)$ for $0 \leqslant i \leqslant n$ and $H(i) = 0$ for $n+1 \leqslant i \leqslant d+1$. A simplicial complex $\Sigma$ is constructed by choosing as its maximal simplices certain $(d+1)$-sets from a $\nu$-set, where $\nu = H(1) + d + 1$, such that $\Sigma$ is shellable in the sense of [9] and such that $(H(0), \ldots, H(d+1))$ is its $h$-vector. It is then shown that $\Sigma$ is the complex associated with a shellable proper collection $\mathcal{B}$ of facets of the cyclic polytope $C(\nu, d+1)$, implying that $|\Sigma|$ is a $d$-ball. $\partial|\Sigma|$ is then a $(d-1)$-sphere with associated complex $\partial\Sigma$. From $H(i) = 0, n+1 \leqslant i \leqslant d+1$, it can be concluded that $h_i^{(d+1)}(\partial\Sigma) = h_i^{(d+1)}(f), 0 \leqslant i \leqslant n$. This and the Dehn-Sommerville equations for $\partial\Sigma$ yield $h^{(d+1)}(\partial\Sigma) = h^{(d+1)}(f)$, whence we conclude $f = f(\Sigma)$. Next, with an appropriate realization of $C(\nu, d+1)$ in $\mathbf{R}^{d+1}$, a point $z \in \mathbf{R}^{d+1}$ can be found such that $z$ is beyond those facets of $C(\nu, d+1)$ that are in $\mathcal{B}$ and beneath the rest. Then the vertex figure $P$ of $z$ in $\operatorname{conv}(C(\nu, d+1) \cup \{z\})$ is a $d$-polytope whose boundary complex is isomorphic to $\partial\Sigma$, demonstrating sufficiency. (In fact, $P$ is $n$-stacked in the sense of [7].)

A sketch of the construction of $\Sigma$ follows. The case $H(1) = 0$ is easily dealt with. For $H(1) \geqslant 1$, let $U = \{u_1, \ldots, u_{\nu'}\}$ where $\nu' = H(1) + 2n$. Let $\Psi'$ be the set of all $2n$-subsets $W'$ of $U$ of the form $\{u_{i_1}, u_{i_1+1}\} \cup \cdots \cup \{u_{i_n}, u_{i_n+1}\}$ where $1 \leqslant i_1, i_n + 1 \leqslant \nu'$, and $i_{j+1} > i_j + 1$, $1 \leqslant j \leqslant n-1$. Let $V' = \{v_1, \ldots, v_{d+1-2n}\}$, $V = V' \cup U$ and $\Psi$ be the set of all $(d+1)$-subsets $W$ of $V$ of the form $V' \cup W'$ for $W' \in \Psi'$. Give the elements of $\Psi$ the lexicographic linear order $<$ induced by $u_1 < \cdots < u_{\nu'}$. Let $\Phi_n$ be the set of all monomials in the variables $Y_1, \ldots, Y_s$ of degree at most $n$, where $s = H(1)$. A one-to-one order preserving correspondence $\beta: \Phi_n \longrightarrow \Psi$ can be defined. From $\Phi_n$ choose the lexicographic order ideal of monomials $M$ associated with $(H(0), \ldots, H(n))$. List the elements of $M$ in order $m_1 < \cdots < m_\mu$. Consider the corresponding elements of $\Psi$, $F_i = \beta(m_i)$. Let $\Sigma$ be the $d$-dimensional simplicial complex whose maximal simplices are $F_1, \ldots, F_\mu$. It can be shown that $\Sigma$ is shellable with shelling order $F_1, \ldots, F_\mu$ and that $h^{(d+1)}(\Sigma) = (H(1), \ldots, H(d+1))$.

Relabel the elements of $V = \{v_1, \ldots, v_{d+1-2n}, u_1, \ldots, u_{\nu'}\}$ as $\{v_1, \ldots, v_\nu\}$ where $\nu = H(1) + d + 1$. Consider the cyclic polytope $C(\nu, d+1) = \operatorname{conv}\{v_1, \ldots, v_\nu\}$ where $v_i = (t_i, t_i^2, \ldots, t_i^{d+1}) \in \mathbf{R}^{d+1}$, $t_1 < \cdots < t_\nu$. This notation implicitly defines a one-to-one correspondence between $V$ and the vertex set of $C(\nu, d+1)$. Then $\{F_1, \ldots, F_\mu\}$ is a representation of a shellable proper collection $\mathcal{B}$ of facets of $C(\nu, d+1)$. The existence of a realization of $C(\nu, d+1) \subseteq \mathbf{R}^{d+1}$ and of a point $z \in \mathbf{R}^{d+1}$ beyond precisely the facets in $\mathcal{B}$ reduces to finding rational numbers $t_1 < \cdots < t_\nu$ satisfying a finite number of polynomial inequalities. This can be accomplished by an application of a version of Tarski's Principle (see e.g. [2, Theorem 13, p. 290]). Once this is done, the desired simplicial polytope $P$ can be obtained as previously described.

A PROBLEM OF KLEE ON UPPER BOUNDS. For $3 \leqslant d \leqslant r < \nu$, a polytope (resp. spherical complex) $P$ is of type $(d, \nu, r)$ if $P$ is a $d$-polytope (resp. $(d-1)$-spherical complex) with $\nu$ vertices, one of which is incident to precisely $r$ edges. The problem, stated by Klee in a dual fashion, is to determine $\max f_{d-1}(P)$ over all simplicial polytopes $P$ of type $(d, \nu, r)$. Klee places bounds on this number and determines it in some particular cases [3], [4]. We offer the complete solution with the following

THEOREM 2. *Let $S$ be a simplicial sphere of type $(d, \nu, r)$. Then* $f_i(S) \leqslant f_i(C(\nu-1, d)) + f_i(C(r+1, d)) - f_i(C(r, d))$, $0 \leqslant i \leqslant d-1$. *Further, there exists a simplicial $d$-polytope $P^*$ that satisfies all of the above expressions with equality. (Here $f_i(C(d, d))$ is 2 if $i = d-1$, and is $f_i(C(d, d-1))$ otherwise.)*

The bounds are established in the same manner that Stanley uses in [8], relying on the fact that $h$-vectors of simplicial spheres are 0-sequences. $P^*$ is obtained from a construction similar to that used in the proof of Theorem 1. Here, however, the desired polytope is $\operatorname{conv}(C(\nu-1, d) \cup \{z\})$ for an appropriate $z$. By a triangulation argument similar to that of pulling vertices of polytopes it can in fact be shown that $P^*$ achieves the maximum number of $i$-dimensional faces over the class of all (not necessarily simplicial) spherical complexes of type $(d, \nu, r)$. (Spherical complexes are defined in [6].)

## REFERENCES

1. B. Grünbaum, *Convex polytopes*, Wiley, New York, 1967.

2. N. Jacobson, *Lectures in abstract algebra*, Vol. III, Van Nostrand, Princeton, N. J., 1964.

3. V. Klee, *Polytope pairs and their relationship to linear programming*, Acta Math. 133 (1974), 1–25.

4. ———, *Convex polyhedra and mathematical programming*, Proc. Internat. Congr. Mathematicians, Vol. I, Vancouver, 1974, pp. 485–490.

5. P. McMullen, *The numbers of faces of simplicial polytopes*, Israel J. Math. 9 (1971), 559–570.

6. P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, London Math. Soc. Lecture Note Ser. 3, Cambridge, 1971.

7. P. McMullen and D. W. Walkup, *A generalized lower-bound conjecture for simplicial polytopes*, Mathematika **18** (1971), 264–273.

8. R. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Appl. Math. **54** (1975), 135–142.

9. ———, *Cohen-Macaulay complexes*, Higher combinatorics, M. Aigner (ed.), D. Reidel, Dordrecht, 1977, pp. 51–62.

10. ———, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978), 57–83.

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