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A TOPOLOGICAL RESOLUTION THEOREM

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We prove a topological analogue of the resolution theorem for algebraic varieties [**H**]. We show that every compact P.L. manifold M admits a framed stratification (every stratum has a product neighborhood) such that after a sequence of topological blow ups performed along the closed smooth strata we get a compact smooth manifold $\widetilde{M}$ ($\partial\widetilde{M} = \varnothing$ if $\partial M = \varnothing$) and a degree one map (with $Z/2$ coefficients) $\pi: \widetilde{M} \longrightarrow M$. The map π is a P.L. homeomorphism in the complement of a union of smooth submanifolds of the form $N_i \times W_i$, such that π collapses $N_i \times W_i$ to N_i in some order. This structure can be used to show that every compact P.L. manifold is P.L. homeomorphic to a real algebraic variety [**AK**]. This also gives a nice way of defining differential forms on P.L. manifolds by pushing down the relative forms from the smooth resolution spaces.

Define an A_0-structure on a P.L. manifold to be a smooth structure, and call such manifold an A_0-manifold. Inductively define an A_k-structure on a P.L. manifold M to be a decomposition

$$M = M_0 \cup_\phi \coprod_{i=1}^{r} N_i \times \operatorname{cone}(\Sigma_i)$$

for some r, where M_0 is an A_{k-1}-manifold with boundary; each Σ_i is a boundary of a compact A_{k-1}-manifold and is P.L. homeomorphic to a P.L. sphere; and N_i are smooth manifolds. Finally $\phi = \{\phi_i\}$ are maps describing the identification (as stratified sets) $\phi_i: N_i \times \Sigma_i \longrightarrow \partial M_0$ where the union is taken. We say M has an A-structure if it has an A_k-structure for some k.

To describe the blowing up process, let M be an A_k-manifold. Then $M = M_0 \cup \coprod_i N_i \times \operatorname{cone}(\Sigma_i)$ and we can choose compact A_{k-1}-manifolds W_i with $\partial W_i = \Sigma_i$. Construct the obvious A_{k-1}-manifold $\widetilde{M}_{k-1} = M_0 \cup \coprod_i N_i \times W_i$. There is the obvious P.L. map $\pi: \widetilde{M}_{k-1} \longrightarrow M$ which is the identity on M_0 and collapses each $N_i \times W_i$ onto N_i. We can iterate this process to get a resolution sequence

$$\widetilde{M} = \widetilde{M}_0 \xrightarrow{\pi} \widetilde{M}_1 \xrightarrow{\pi} \cdots \xrightarrow{\pi} \widetilde{M}_{k-1} \xrightarrow{\pi} M.$$

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$\widetilde{M}$ is smooth and clearly the composition map $\pi: \widetilde{M} \longrightarrow M$ collapses $N_i \times W_i$ onto N_i.

Following [L] and [W] we define A-thickenings; the classifying space B_A; and the natural map $B_A \longrightarrow B_{PL}$. Then we prove the usual structure theorem: Namely that a compact P.L. manifold M has an A-structure if and only if the normal bundle map (thickening map)

$$M \xrightarrow{\nu M} B_{\mathrm{PL}}$$

lifts to B_A. Let PL/A be the homotopy theoretical fibre of $B_A \longrightarrow B_{PL}$.

THEOREM. *$B_A \simeq B_{PL} \times PL/A$, and PL/A is the product of Eilenberg-Mc Laine spaces $K(Z/2, n)$'s. The number δ_n of $K(Z/2, n)$ for each n in this product is given to be*

$$\delta_n = \begin{cases} 0 & \text{if } n < 8, \\ 26 & \text{if } n = 8, \\ \text{infinite but countable} & \text{if } n > 8. \end{cases}$$

COROLLARY. *Every compact P.L. manifold M has an A-structure and number of different A-structures (up to A-concordance) on M is given by $\bigoplus_{n \geq 8} H^n(M: \pi_n(PL/A))$.*

Roughly an A structure on M gives a topological resolution on M and $\bigoplus_n H^n(M: \pi_n(PL/A))$ classifies different ways of resolving M.

OUTLINE OF PROOF. While constructing B_A we also classify A_k-thickenings and prove the usual classification theorem for them. We then proceed to analize B_{A_k}, and $B_A = \lim_{k\to\infty} B_{A_k}$. It is standard that $\pi_i(PL/A_k)$ coincides with concordance classes of A_k-structures on S^i. Since $\pi_i(PL/A) = \lim_{k\to\infty}\pi_i(PL/A_k)$ it follows from definitions that $\pi_i(PL/A)$ maps monically to the i-dimensional unoriented A bordism group η_i^A. Next we construct a Thom spectrum, MA, with $\pi_i(MA) \approx \eta_i^A$.

Since it is clear that an A_k-manifold crossed with a smooth manifold is an A_k-manifold, we can show that MA is an MO module spectrum and it is now a formality that the map $\eta_i^A \longrightarrow H_i(B_A; Z/2)$ given by

$$\{M \xrightarrow{\nu_M} B_A\} \rightsquigarrow (\nu_M)_* [M]$$

is monic. Hence $\pi_*(PL/A) \longrightarrow H_*(B_A; Z/2)$ is monic, hence split. It is now easy to show that PL/A is a product of $K(Z/2, n)$'s and to construct a map $B_A \longrightarrow PL/A$ splitting the inclusion $PL/A \longrightarrow B_A$.

To compute δ_n is not so hard. To construct B_A one proves along the way that δ_n is countable. For $n \leq 8$ one can explicitly see all the A-spheres so there is no problem. Above 8 things are complicated but it is not too hard to construct an infinite number of concordance classes in each dimension. The details will appear elsewhere [AT].

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