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Locally solid Riesz spaces, by Charalambos D. Aliprantis and Owen Burkinshaw, Academic Press, New York, 1978, xii + 198 pp.

Vector lattices, also called Riesz spaces, have been objects of mathematical interest at least since F. Riesz's pioneering paper [34] at the International Mathematical Congress held at Bologna in 1928. Since then many others have developed the subject. Some of the more important contributions to the theory through 1950 were made by the following authors. H. Freudenthal [14], S. W. P. Steen [37], L. V. Kantorovich [19], M. H. Stone [38], H. Nakano [26], [27], [28], [29], [30], [31], [32], F. Maeda and T. Ogasawara [25], [33], K. Yosida [40], [41], [42], H. F. Bohnenblust [9], S. Kakutani [17], [18].

In the next fifteen years vector lattices were not given much attention. Some important things were done. A paper of I. Amemiya [1] gave many new advances in the algebraic theory, some of which are still being rediscovered. W. A. J. Luxemburg and A. C. Zaanen were also very active at this time with a succession of important papers [22], [23].

In the last decade codification of the theory has, perhaps, begun. Books devoted exclusively or mostly to vector lattices have been written. We have those by W. A J. Luxemburg and A. C. Zaanen [24], H. H. Schaefer [35], D. H. Fremlin [12] and the book under review.

Let us now briefly describe a part of the theory of vector lattices which is of current interest in analysis.,A vector lattice for our purposes is a real vector space L, which is also a lattice, and is subject to the compatability relation $0 \leqslant x, y \in L$, $0 \leqslant \alpha$, $\beta \in \mathbf{R}$ imply $\alpha x + \beta y \geqslant 0$. One has a unique Jordan decomposition $x = x^+ - x^-$ with $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $x^+ \wedge x^- = 0$. The analyst imposes a norm subject to $\|y\| \leqslant \|x\|$ whenever $|y| \leqslant |x|$ ($|y| = y^+ + y^- = y^+ \vee y^- = y \vee (-y)$). When L is complete under its norm we have a Banach lattice.

Linear maps between Banach lattices which are also lattice homomorphisms are somewhat rare. One may ask, however, for a given linear map, isomorphism, or isometry between Banach lattices what can be discovered which is a consequence of the lattice structure. A special case of this is to characterize the ranges of contractive projections on L_p-spaces ($1 \leqslant p < \infty$). These turn out to be precisely the subspaces which are themselves isometric to L_p-spaces (perhaps for a different measure). If $p \neq 2$, they are all obtainable from closed linear sublattices of the original L_p-space by multiplication by a measurable function of absolute value one (modulo unimportant measure theoretic technicalities). This theory goes back to A. Grothendieck [15] and is studied in [11], [3], [39]. A reasonably definitive treatment is given in [6].

Another problem is to decide when a closed subspace of a Banach lattice is linearly isomorphic to a Banach lattice or better, after a multiplication type isometry as above, is in fact a sublattice as well. This problem is essentially open.

The closed subspace problem comes up in the study of local unconditional structure. Any Banach lattice has the property that each of its finite dimensional subspaces can be enlarged to a finite dimensional subspace which is linearly isomorphic to a Banach lattice (finite dimensional of course). We cannot achieve isometry but can make the isomorphism constants tend to one as the dimension tends to infinity. Suppose conversely that one has a Banach space with this type of local (i.e. finite dimensional) behavior. It is easy to show that the space is linearly isometric to a closed subspace of a Banach lattice. (If the isomorphism constants are merely bounded, instead of tending to one, we have an isomorphism rather than an isometry.) With heavy technical hypotheses one can then show that the original space is indeed linearly isometric to a Banach lattice. The case when all the finite dimensional lattices are l_p-spaces (p fixed $1 \leqslant p < \infty$) is easier and one has, in fact, an L_p-space. Whether this last result holds in the isomorphic case with bound sufficiently close to one is not known. H. Elton Lacey [20] gives a good summary of much of this material. The easiest exposition of the general case (local characterisation of Banach lattices with order continuous norm) will appear in [7].

One more current topic can be described briefly as l_p-structure in Banach lattices. This is covered in some detail by J. Lindenstrauss and L. Tzafriri

[21]. Suppose we have a function f of n real variables made up of operations of addition, scalar multiplication, maximum, and minimum. It makes sense to talk of $f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ now come from a vector lattice. Our function f is positively homogeneous of degree 1 ($f(\lambda x_1, \ldots, \lambda x_n) = \lambda f(x_1, \ldots, x_n)$ for all $\lambda \geqslant 0$). In a Banach lattice we can extend from this subclass of functions to define $f(x_1, \ldots, x_n)$ for any positively homogeneous function f of degree one. In particular we can define for $1 \leqslant p < \infty$ and $x_1, \ldots, x_n$ in a Banach lattice $(\Sigma_{i=1}^n |x_i|^p)^{1/p}$. If there is a two-sided estimate for the norm of all such elements in terms of $(\Sigma_{i=1}^n \|x_i\|^p)^{1/p}$ we have an isomorphic characterisation of L_p-spaces. By assuming one-sided estimates we have the notions of p-convexity or p-concavity. By restricting to $x_1, \ldots, x_n$ such that $|x_i| \wedge |x_j| = 0$ $(i \neq j)$ we have the weaker notions of upper and lower p-estimates. These link up with ideas of type and cotype which are defined in general Banach spaces. Very roughly a Banach lattice with a lower q-estimate has cotype q, and type r implies p-convex for $1 < p < r$.

Despite this activity the study of vector lattices still has a somewhat uneasy role in mathematics. In its purely algebraic aspects, lattice and linear structure, analysts largely ignore it. Bourbaki [10] has one short section on the subject. J. Lindenstrauss and L. Tzafriri [21] prefer to avoid algebraic ideas entirely and do so by a devious route through a concrete representation of free vector lattices and the full force of the representation theory for M-spaces [18]. In fact this is not too unreasonable in their context. Some extra order algebra would avoid the M-space results but not the free vector lattices. (They never actually use the term free vector lattice.) This method also gives the homogeneous function results mentioned above.

Probably the main reason for treating vector lattices in such cavalier fashion is the extreme simplicity of the lattice axioms. The classical spaces L_p, $C(K)$ are all vector lattices, but coming as they do, with natural concrete representations as function spaces the consequences of the lattice structure are more easily seen as consequences of this representation. Indeed since the free vector lattice on a finite number of generators has a concrete representation as a function lattice [8], any vector lattice identity valid in the reals is valid in an arbitrary vector lattice. There are therefore many results which the working analyst can prove for himself, as they are needed, without bothering to develop, or consult the general theory. This leads to much duplication. One example of this can be found in the discovery of the complete Boolean algebra of polar subspaces made by F. Maeda and T. Ogasawara [25] as part of their representation theory. Among those who have rediscovered this result are F. Sik [36], J. Isbell [16], and the reviewer [4].

The other side of the coin shows when topological structure is imposed as well. Even in the simplest cases, normed vector lattices, Banach lattices, the relationships between order and topology do not have the far reaching consequences we find in, say, Banach algebra theory. Order completeness is independent of topological completeness and order convergence of topological convergence. The lattice operations may be continuous at the origin, without being uniformly continuous. Linear transformations may be continuous without respecting order bounded sets. Continuous linear functionals may ignore order convergence of nets or even of sequences. Order continuity for nets is independent of order continuity for sequences. We thus have a topological dual space, an order dual, an order sequentially continuous dual (integrals) and an order continuous dual (normal integrals) all of which are important. There are, seemingly endless, technical byways in which to lose oneself.

Coupled with this we have no general agreement over terminology, even for the basic concepts, nor for notation as Fremlin [13] points out in his review of [35].

The books we have mentioned are widely different in character. Luxemburg and Zaanen [24] give in their volume I, an exhaustive treatment of the algebraic side of the theory. They go into great detail with related subjects (Boolean algebra, distributive lattices). They present all the representation theories for archimedean vector lattices, and show how to obtain some classical results of analysis (Radon-Nikodym theorem, spectral theorem in Hilbert space, Poisson formula) by these methods. Topology is never imposed and they leave linear functionals to volume II. Schaeffer's book [35] is probably the most useful. It treats Banach lattices in some depth, and does not get too heavily involved in peripheral details. Fremlin's book [12] is never dull. It contains much that is elegant and new. The approach is topological linear space rather than norm and the style idiosyncratic.

Now we consider the book under review. In approach it most resembles Fremlin's book. Some of Fremlin's results are given, with slightly easier proofs. The style is less flamboyant. Unfortunately the authors have followed Fremlin for terminology so we find much use of a proper name as the adjective to describe one property or another. They have also left out all motivations, applications and connections with other branches of analysis. Their choice of topics seems guided by their own research interests. Much of the new material on locally convex solid spaces carries over naturally from the Banach lattice situation. Apart from the intrinsic interest in these generalizations it would have been good to have concrete evidence of their usefulness. There are many improvements on existing proofs, and plenty of material to interest experts on topological vector lattices. The more casually interested reader may find less to attract him.

There are seven chapters, each supplemented by exercises. Open problems are given after all but the first two chapters. Chapter I is introductory and contains the standard results about vector lattices, homomorphisms, ideals, quotients, order completeness (here called Dedekind completeness), projection properties, order bounded linear maps, order duals, order convergence and also some standard linear topological space material. Chapter 2 imposes compatibility between order and topology. The requirement of uniform continuity for the lattice operations leads to locally solid topologies. A solid set in a vector lattice is a set A such that $|x| \leq |y|$ and $y \in A$ imply $x \in A$; a locally solid topology is a linear space topology where the zero element has a base of solid neighborhoods. Locally solid and Hausdorff implies archimedean. With local convexity added, duality theory is developed and the topology is generated by a family of monotone seminorms. There is also a section on topological completion.

Chapter 3 is concerned with order continuity. An order continuous topology is a locally solid topology such that order convergence implies topological convergence. The authors follow Fremlin in calling this a Lebesgue topology. (Other terms in the literature for this property are condition (A, ii) and universally continuous.) There are also σ-order continuous topologies for which order convergent sequences are topologically convergent, and pre order continuous topologies for which order bounded increasing sequences (and hence nets) are Cauchy. These notions are investigated and characterizations and interrelations given. A section on L_p-spaces gives the characterisations of H. F. Bohnenblust [**9**] and S. Kakutani [**17**], [**18**].

Chapter 4 considers order closed topologies, those with a base of neighborhoods of zero consisting of solid order closed sets. Again following Fremlin, these are called Fatou topologies. They are shown to lift to the Dedekind completion and, when Hausdorff, to have a finer restriction to order bounded sets than any Hausdorff order continuous topology. The deep theorem of Nakano [**32**] that order intervals are topologically complete for any order complete vector lattice with an order closed topology is proved. The proof given here, and that of Amemiya [**2**], are the two best proofs of this result. The proper name soup continues with: Levi space (for boundedly order complete), meaning every topologically bounded increasing directed set has a supremum; Nakano space, one which is Levi and Fatou; and the theorem, every Nakano space is topologically complete.

Chapter 5 considers metrizable locally solid spaces with particular attention to order continuity properties.

Chapter 6 considers duality for convex locally solid vector lattices. Most of the chapter is concerned with weak compactness in the order dual and characterizing compact solid sets by disjoint sequences. This chapter is probably the best in the book.

Chapter 7 contains some results about laterally complete vector lattices. (Laterally complete means that every nonempty pairwise disjoint subset has a supremum.) Topologically this is an unnatural condition. For the normed case even σ-lateral completeness forces finite dimensonality. Consequences of lateral completeness, such as Archimedean plus (σ-) laterally complete imply the (principal) projection property [**5**] are obtained and the independence of lateral completeness and order completeness is shown. Finally some topological consequences of lateral completeness are considered. Roughly, there is at most one Hausdorff order closed topology on a σ-laterally complete vector lattice and if there is one, it is necessarily order continuous.

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Mathematical logic. An introduction to model theory, by A. H. Lightstone, Mathematical Concepts and Methods in Science and Engineering, Vol. 9, Plenum Press, New York and London, 1978, xiii + 338 pp.,$22.50.

1. In the past twenty years or so mathematical logic has moved from being a subject often considered rather exotic (if indeed it was really mathematics) to being a subject about which most mathematicians ought to know at least a little. The reasons are not hard to find. First, mathematical logic essentially enshrines the idea of precision in mathematical language. Second, it treats of the logical processes of deduction and makes clearer the abstract structure of arguments. Third, the techniques involved lead to new developments in and of other parts of mathematics.

Precision of language was encouraged, even demanded, by the nineteenth century crises in analysis and, later, set theory. (How easy it is now to distinguish between convergence: $\forall \varepsilon > 0 \ \forall x \ \exists \delta > 0 \ldots$ and uniform convergence $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ldots$ How tricky for Cauchy.)

The analysis of deduction culminates in the provision of a neat (essentially finite) presentation of axioms and rules which give only true statements and, in certain cases, all true statements (the completeness theorems).

New techniques emerged including the ideas of recursive functions and the development of computer programs. (These together with the precision of language led to an unexpected answer to e.g. Hilbert's tenth problem: there is no general formal technique which will decide diophantine problems.) Recursive function theory comes from the ideas of formal languages; the other aspect, truth, leads to model theory: the semantic aspect of the languages. Most present general interest here centres on nonstandard analysis. Abraham Robinson's brilliantly simple observation was to apply a reasonably well-known theorem (compactness) in what appeared an entirely unpromising situation.

2. Corresponding to the three aspects of logic noted above (though not in one-one correspondence) are three theorems.

Propositional calculus deals only with logical connectives (e.g. and, or, not) applied to unanalyzed statements and is very useful as a pedagogical prelude. The first theorem (completeness of propositional calculus) shows that a finite