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Convexity in the theory of lattice gases, by Robert B. Israel, Princeton Series in Physics, Princeton Univ. Press, Princeton, N. J., 1979, LXXXV + 168 pp., $16.50 (cloth).

Lattice gases? This sounds very much like physics. And what have they to do with convexity? The mathematician may be pardoned if he is puzzled, but he could't do better than to look into this book if he wants to find out what this is all about. Lattice gases are certain mathematical models that occur in statistical mechanics. Statistical mechanics was created a near-century ago by J. Willard Gibbs, who conceived the idea of a general explanation for the laws of thermodynamics. It was also Gibbs who in two pioneering papers, rather neglected ever since, suggested that the proper general formulation of the laws of thermodynamics may be made in terms of certain functions, called thermodynamics potentials, which characterize the physical systems considered, and whose *convexity* is the mathematical expression of the stability of states of thermal equilibrium. Our book under review is actually two books in one; the first is an introductory essay by Arthur Wightman, which contains the historical motivation, an exposition of the Gibbsian ideas, the significance of convexity of the thermodynamic potentials, as well as a brief review of the formalism of statistical mechanics as left to us by Gibbs. This is far more than an introduction, and it alone is worth the price of the book. The reader is advised to come back to it from time to time, when studying the more technical proofs of Israel's chapters, to gain motivation, deepen understanding, and appreciate interconnections.

On to the technicalities. First, definitions. A lattice gas is a mathematical system determined by five things, ν, Ω_0, μ_0, Ω and $\mathfrak{B}$. ν is a positive integer, the "dimension". Ω_0 is a compact Hausdorff space, frequently just a finite set. μ_0 is a distinguished natural normalized measure on Ω_0, e.g. Haar measure if Ω_0 is a group, uniform surface measure if Ω_0 is a sphere, normalized counting measure if Ω_0 is finite. With $\mathbf{Z}$ the set of integers, write $L = \mathbf{Z}^\nu$ (the "lattice"), and think of a copy of (Ω_0, μ_0) attached to each point ("lattice site") of L. Ω is defined as a closed, translation-invariant (under the natural action of the additive group L) subspace of the compact space Ω_0^L; thus a point ("configuration") $\omega \in \Omega$ is a function $L \to \Omega_0$ assigning a "coordinate" $\omega_x \in \Omega_0$ to each $x \in L$. A typical example is $\Omega_0 = \{0, 1\}$, $\Omega = \{\omega \in \Omega_0^L\colon\ \omega_x\omega_y = 0$

whenever x, y are nearest neighbors in $L\}$. Finally, $\mathfrak{B}$ is a certain Banach space, whose elements ("interactions") are families $\Phi = (\Phi_X)$ of continuous functions $\Phi_X\colon \Omega \to \mathbf{R}$, indexed by finite subsets X of L. They are subject to three conditions: (a) $\Phi_X(\omega)$ depends on ω only through the coordinates ω_x with $x \in X$, (b) $\Phi_X(\omega)$ is invariant under the same translation of both X and ω, and (c) $\|\Phi\| = \Sigma\{|X|^{-1} \sup|\Phi_X|\colon X \text{ contains the origin}\} < \infty$. One should think of a particular $\Phi \in \mathfrak{B}$ specifying a particular model with all its external parameters fixed; as Φ varies over some finite dimensional subspace $\mathfrak{N}$ of $\mathfrak{B}$, one thinks of a thermodynamic phase diagram with a certain number of variable parameters. Incidentally, the idea to consider not just a finitely many-parameter family of interactions, but the whole space $\mathfrak{B}$, is a rewarding one, shown by the interesting general theorems obtained; still it is advisable for motivational purposes to think of Φ restricted to a small subspace. The heuristic interpretation of Φ is this: one imagines a contribution to the energy of a configuration ω to come from each part (restriction) to finite subsets X, this is just $\Phi_X(\omega)$. The sum $A_\Phi(\omega) = \Sigma\{|X|^{-1}\Phi_X(\omega)\colon 0 \in X\}$ is then the contribution of one lattice site (the origin) to the energy. In statistical mechanics a state of the lattice gas is defined to be a probability measure ρ on Ω; of primary interest are translation-invariant states, corresponding to the idea of a spatially homogeneous condition throughout L. The set of translation-invariant states ρ on Ω is convex and weak*-compact; its extremals are called ergodic states. For a translation-invariant state ρ the integral, or expectation value, $\int A_\Phi \, d\rho$ is the mean energy per lattice site.

There are two important functionals s and P ("entropy" and "pressure"), in some sense duals of each other. s is defined over the set of translation-invariant states, $s = s(\rho) = \lim\{-|\Lambda|^{-1} \int \rho_\Lambda \log \rho_\Lambda \, d\mu_0^\Lambda\}$, where ρ_Λ is the Radon-Nikodym derivative of ρ with respect to the natural product measure μ_0^Λ over configurations in the finite set Λ, and the limit is taken with increasing $\Lambda \to L$. $s(\rho)$ is affine and weak*-upper semicontinuous. This entropy notion is closely related to that introduced into ergodic theory by Kolmogoroff and Sinai. The pressure functional is defined over $\mathfrak{B}$ by

$$P = P(\Phi) = \lim\left(|\Lambda|^{-1} \log \int \exp\left(-\sum\{\Phi_X(\omega)\colon X \subseteq \Lambda\}\right) d\mu_0^L\right),$$

the limit as before. $P\colon \mathfrak{B} \to \mathbf{R}$ is nonincreasing, convex, and Lipschitz-continuous $|P(\Phi) - P(\Psi)| \leqslant \|\Phi - \Psi\|$. The duality of P and s is stated in the fundamental theorem ("Variational Principle") to the effect that the supremum of $s(\rho) - \int A_\Phi \, d\rho$ with respect to translation-invariant states ρ is $P(\Phi)$, and the infimum of $P(\Phi) + \int A_\Phi \, d\rho$ with respect to $\Phi \in \mathfrak{B}$ is $s(\rho)$. But more is true. An $\alpha \in \mathfrak{B}^*$ (the dual space of $\mathfrak{B}$) is said to be P-bounded if $P(\Phi) - \alpha(\Phi)$ is bounded below on $\mathfrak{B}$; it is said to be tangent to P at Ψ (a subdifferential of P at Ψ) if the infimum is actually an assumed minimum at $\Phi = \Psi$.

THEOREM. *If $\alpha \in \mathfrak{B}^*$ is P-bounded, then for each $\Phi \in \mathfrak{B}$ there is a unique translation-invariant state ρ such that $\alpha(\Phi) = -\int A_\Phi \, d\rho$. If, in addition, α is tangent to P at Φ, then the corresponding ρ also satisfies $s(\rho) = P(\Phi) + \int A_\Phi \, d\rho$ (the case of equality in the Variational Principle). Conversely, if Φ, ρ give* *equality in the Variational Principle, then* $\alpha \in \mathfrak{B}^*$ *defined by* $\alpha(\Psi) = -\int A_\Psi \, d\rho$ *is tangent to* P *at* Φ. (In this case ρ is called a translation-invariant equilibrium state for the interaction Φ.)

In general, there may be more than one equilibrium state corresponding to a given interaction, but in any case there is at least one.

Equilibrium states can be analyzed in greater detail when the interaction belongs to a certain subclass $\tilde{\mathfrak{B}}$ of $\mathfrak{B}$ determined by the condition $\Sigma\{\sup|\Phi_X|: \ 0 \in X\} < \infty$. In that case, an equilibrium state ρ satisfies a system of linear equations first formulated by Lanford, Ruelle, and, independently, Dobrushin (the DLR equations). The precise statement is somewhat technical and is omitted here; be it enough to say that it is the precise version of the Boltzmann-Gibbs wisdom: The probabilities of different configurations are in inverse proportion to the exponential function of the corresponding energies. But I cannot resist a beautiful theorem. Two interactions $\Phi, \Psi \in \tilde{\mathfrak{B}}$ are called physically equivalent if a certain linear transformation S annihilates the difference $\Phi - \Psi$. S is defined by the double sum

$$(S\Phi)_X = \sum \left\{ \sum \left\{ (-1)^{|Y \cap Z^c|} \int \Phi_Y \, d\mu_0^{Z^c} : Z \subseteq Y \right\} : Y \supseteq X \right\}.$$

THEOREM. *For* $\Phi, \Psi \in \tilde{\mathfrak{B}}$ *the following four conditions are equivalent*: (i) Φ, Ψ *are physically equivalent*, (ii) P *is linear on the line segment* $\Phi\Psi$, (iii) *there is a translation-invariant state* ρ *satisfying the DLR-equations for both* Φ *and* Ψ, *and* (iv) *every translation-invariant state* ρ *satisfying the DLR-equations for* Φ *also does so for* Ψ. (*This justifies the term "physical equivalence"*.)

The set of states, not necessarily translation-invariant, satisfying the DLR-equations for a given Φ is convex and weak*-compact; what about its extreme points?

THEOREM. *Every state in this set is the unique convex combination* (*in the sense of Choquet*) *of extremals.*

THEOREM. *A state in this set is extremal precisely when it has "short range correlations".*

A state ρ has this property if for every f: $\Omega \to \mathbf{R}$ continuous, $\sup|f| \leqslant 1$, and $\epsilon > 0$, there is a sufficiently large finite subset Λ of L such that if g: $\Omega \to \mathbf{R}$, $\sup|g| \leqslant 1$, and $g(\omega)$ depends only on the ω_x with $x \notin \Lambda$, then $|\int fg \, d\rho - \int f \, d\rho \int g \, d\rho| \leqslant \epsilon$. Another form of this property: If f: $\Omega \to \mathbf{R}$ is bounded, nonconstant, there is some finite Λ such that if g: $\Omega \to \mathbf{R}$ depends only on ω_x with $x \notin \Lambda$, then $\rho\{\omega \in \Omega: f(\omega) \neq g(\omega)\} > 0$.

What about the inverse problem of finding an interaction for which a given state is an equilibrium state? The somehwhat surprising theorem is proved that, if $\rho_1, \rho_2, \ldots, \rho_k$ are arbitrary ergodic (i.e., extremal translation-invariant) states with finite entropy, then there is an interaction $\Phi \in \mathfrak{B}$ for which not only one but all of them are equilibrium states. This theorem induces the author to declare that while the Banach space $\mathfrak{B}$ is in some respects very natural, it is too large and a source of pathology. A further example of this pathology: There is in $\mathfrak{B}$ a dense set of interactions which have uncountably many ergodic equilibrium states. Fortunately, all is not as bad as that. For one thing, much of the pathology disappears if one restricts oneself to the smaller space $\tilde{\mathfrak{B}}$ of interactions. For another thing, there is the classic theorem of Mazur, according to which the set of points Φ at which P has a unique tangent is a dense G_δ-set in $\mathfrak{B}$, so from the point of view of Baire-category the occurrence of more than one ergodic equilibrium state for the same interaction is an exceptional phenomenon. The last chapter of our book discusses a strengthening of this statement. Think of some finite dimensional subspace $\mathfrak{N}$ of $\mathfrak{B}$. If $\mathfrak{N}$ is well behaved, one expects the majority of points $\Phi \in \mathfrak{N}$ to possess exactly one equilibrium state. Exceptionally, there may be two ergodic equilibrium states, but the subset of $\mathfrak{N}$ where this is the case should be small. Even smaller should be the subset of $\mathfrak{N}$ where there are three, and so on. Let us say that the Gibbs Phase Rule holds in $\mathfrak{N}$ if the set $\{\Phi \in \mathfrak{N}\colon \Phi$ has k ergodic equilibrium states$\}$ has Hausdorff-dimension at most $n - k + 1$ in $\mathfrak{N}$, where $n = \dim(\mathfrak{N})$. The set of n-dimensional subspaces of $\mathfrak{B}$ can be made into a complete metric space $\mathcal{G}_n$. The precise version of the statement "The Gibbs Phase Rule holds generically" is the theorem that $\{\mathfrak{N} \in \mathcal{G}_n\colon$ The Gibbs Phase Rule holds in $\mathfrak{N}\}$ is a dense G_δ-set in $\mathcal{G}_n$. This is quite satisfactory, although conceivably one might want to know more, for instance, if $\dim(\mathfrak{N}) = 2$ and $k = 3$ ("triple point" or coexistence of three pure phases), one expects only isolated points in $\mathfrak{N}$, not merely a set of Hausdorff dimension 0; similarly for $k = 2$ (coexistence of two pure phases) one should have nice curves in some sense or other. Perhaps future research will succeed in this direction.

One more remark on the contents of the book. All definitions and theorems have their quantum mechanical analogues. In fact, the material is so organized that the two cases, classical and quantum, are discussed side by side, so that the investigation proceeds in parallel. In fact, on the level of general theory, there is hardly any difference, and the effect of quantum modification is present only for specific properties of specific models.

My own assessment is that this book is a valuable compendium for research workers in the mathematical aspects of statistical mechanics, and it should also succeed in attracting outsiders from the mathematical community to acquaint themselves with a fascinating topic.

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Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics, by David Ruelle, Encyclopedia of Mathematics and its Applications (Gian-Carlo Rota, Editor), vol. 5, Addison-Wesley, Reading, Mass., 1978, xix + 183 pp., $21.50.

The book under review is concerned with the general aspects of classical equilibrium statistical mechanics of lattice systems and some generalizations. Before commenting on the book we shall describe some of the main *mathematical* issues which arise in this deep and active area of mathematical