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STOCHASTIC INTEGRATORS

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ABSTRACT. The most general reasonable stochastic integrator is a semimartingale. For a large class of integrands the stochastic integral can be evaluated pathwise.

The notion of a semimartingale is a notion *ad hoc.* It is the result of an effort to generalize to the utmost the two known *techniques* of stochastic integration: If $Z = M + V$ is a decomposition of the semimartingale Z into a local martingale M and a process V of finite variation then one can define $\int_0^t X\,dZ$ as $\int_0^t X\,dM + \int_0^t X\,dV$. The second summand is an ordinary Stieltjes integral taken pathwise, while the first one is defined by Ito's technique.

The question arises whether this is the best one can do. More precisely, for which processes Z can one define $\int \cdot\, dZ$ in such a way that the integral has "reasonable" properties? Somewhat disappointingly, the theorem below states that every reasonable stochastic integrator is a semimartingale. This might come as a surprise in view of the very modest criterion of reasonableness adopted.

On the positive side, the proof of the theorem yields an equivalent definition of the integral which obviates the need to split the integrator Z as $Z = M + V$ and which lends itself to a *pathwise* computation of $\int X\,dZ$ for a large class of integrands X.

Stating our criterion of reasonableness requires some notation. Underlying everything is a complete probability space $(\Omega, \mathcal{G}, P)$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t\colon 0 \leq t < \infty)$ that has the usual properties [5], [7]. Let $\mathcal{T}$ denote the collection of all stopping times that take only finitely many values, each of them finite; let $\mathcal{A}$ denote the ring of subsets of the base space $B = \Omega \times [0, \infty)$ generated by the stochastic intervals $((S, T]]$, $S \leq T$ in $\mathcal{T}$; and denote by $\mathcal{R}$ the vector lattice of step functions over $\mathcal{A}$. Now if $Z\colon B \longrightarrow \mathbf{R}$ is any process then $dZ(((S, T]]) := Z_T - Z_S$, extended by linearity, defines a linear map

$$dZ\colon\ \mathcal{R} \longrightarrow L^0(\Omega, \mathcal{G}, P).$$

The following definition spells out our criterion of reasonableness.

DEFINITION. An adapted process Z is an L^p-integrator, $0 \leq p < \infty$, provided

(A_0) if A_n is a decreasing sequence in $\mathcal{A}$ with void intersection then

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$dZ(A_n) \longrightarrow 0$ in L^0 (i.e. in measure), and

(B_p) The set $\{dZ(X)\colon X \in \mathbf{R}, |X| \leqslant 1, X = 0 \text{ on } ((t, \infty))\}$ is bounded in L^p, for every instant $t < \infty$.

A simple upcrossing argument shows that, in the presence of (B_0), (A_0) is equivalent with the existence of a right continuous version of Z with left limits (r.c.1.1.). Therefore, Z will henceforth be assumed r.c.1.1., so that (B_p) is the only additional restriction.

THEOREM 1. *Let Z be an adapted r.c.l.l. process. The following are equivalent.*

(I) *Z is an L^0-integrator (i.e., satisfies B_0));*

(II) *Z is a semimartingale;*

(III) *There is an extension $\int \cdot\, dZ$ of dZ to a class of processes such that the dominated convergence theorem holds.*

The implications (III) $\longrightarrow$ (I) and (II) $\longrightarrow$ (I) are almost evident. The methods of proof of (I) $\longrightarrow$ (III) $\longrightarrow$ (II) have many other applications and are perhaps more interesting than the theorem itself, and so we provide a brief sketch.

(I) $\longrightarrow$ (III). We shall prove the corresponding statement for all $p \in [0, \infty)$ simultaneously. Let ρ_p denote the usual translation invariant metric on L^p, and define a function metric G_p on the set of all processes in Daniell's way: if $H \geqslant 0$ is the supremum of a sequence in $\mathcal{R}$, set $G_p(H) = \sup\{\rho_p(dZ(X))\colon X \in \mathcal{R}, |X| \leqslant H\}$, and for an arbitrary process $F\colon B \longrightarrow \mathbf{R}$ set $G_p(F) = G_p(F; Z, P) = \inf\{G_p(H)\colon |F| \leqslant H, H \text{ as above}\}$. G_p is an upper gauge in the sense of [1] (cf. also [2], where their simple theory is developed in detail). The closure $\mathcal{L}_p = \mathcal{L}_p[Z, P]$ of $\mathcal{R}$ in G_p-mean is a complete space of processes on which the dominated convergence theorem holds, and $\int \cdot\, dZ\colon \mathcal{L}_p \longrightarrow L^p$ is defined as the extension of $dZ\colon \mathcal{R} \longrightarrow L^p$ by continuity. All this is, of course, but a slight extension of ITO's procedure. Despite their down-to-earth definition, the upper gauges G_p are most useful devices.

For the implication (III) $\longrightarrow$ (I) we establish a lemma that has many other applications.

LEMMA. *Let Z be an $L^0(P)$-integrator, T an a.s. finite stopping time, and $0 < q \leqslant 2$. There exists a probability measure P', equivalent with P on $\mathcal{G}$, such that the stopped process Z^T is a (global) $L^q(P')$-integrator.*

Choose, for instance, $q = 2$. Any $L^2(P')$-integrator is easily seen to be a P'-semimartingale and hence a P-semimartingale. We apply this to Z^T and let $T \longrightarrow \infty$.

Here is a brief proof of the lemma. The linear map $dZ^T\colon \mathcal{R} \longrightarrow L^0(P)$ is bounded when $\mathcal{R}$ is given the sup-norm topology; and $\mathcal{R}$ has the same finite-dimensional subspace structure as a space $L^\infty(\mu)$. By a deep factorization theorem of Maurey [6] and Rosenthal [9] there exists a function f in $L^0(P)$ and a continuous linear map U: $\mathcal{R} \longrightarrow L^q(P)$ such that $dZ^T(X) = f \cdot U(X)$ for all $X \in \mathcal{R}$. Set $P' = c/(1 + f^q) \cdot P$, where c is chosen so that $P'(\Omega) = 1$.

The value of the integral is, of course, the same as that arrived at by the procedure outlined in the first paragraph. Note that no splitting $Z = M + V$ has to be found, though. Despite its definition as a limit in p-mean ($0 \leqslant p < \infty$), the integral is *local in nature.* In order to clarify the meaning of this statement, we make a small observation. The process $t \longrightarrow \int_0^t X\,dZ$ is evidently again an L^0-integrator when $X \in L_0[Z; P]$ and so has a r.c.l.l. version, which is unique up to indistinguishability. This version is denoted $X * Z$. It is almost evident from our definition of the integral that when the paths of the pairs (X, Z) and (X', Z') coincide a.s. on some set $\Omega_0 \subset \Omega$ then so do those of $X * Z$ and $X' * Z'$. When the integrand X is left continuous and has no oscillatory discontinuities then the integral $X * Z$ can actually be evaluated *pathwise.* Note that the integrand $F(Y_-)$ in a stochastic differential equation $dY = F(Y_-)dZ$ driven by an L^0-integrator Z is of this description, and that the following theorem actually furnishes an *algorithm* approximating the integral pathwise.

THEOREM 2. *Let Z be a semimartingale and X a r.c.l.l. process whose maximal process $|X|_t^* = \sup\{|X_s|:\ 0 \leqslant s \leqslant t\}$ is a.s. finite at all finite instants t. Let X_- denote the left continuous version of X. Then almost surely $X_- * Z$ is uniformly on bounded intervals the limit of the sequence of processes Y^n given by*

$$Y_t^n = \sum_i X_{T_i^n}(Z_{T_{i+1}^n \wedge t} - Z_{T_i^n \wedge t}),$$

where $T_0^n = 0$ and $T_{i+1}^n = \inf\{t > T_i^n:\ |X_{T_i^n} - X_t| > 2^{-n}\}$. In fact, for any a.s. finite stopping time T

$$\sum_n |Y^{n+1} - Y^n|_T^* < \infty \quad a.s. \tag{1}$$

In other words, one can approximate the integral by Riemann sums, where the partition is chosen not according to the values of the integrand but according to the variation of its values. The argument will show that some errors are permissible. Essential is only that the T_i^n are so chosen that $\max_i |X - X^{T_i^n}|^*_{T_{i+1}^n-}$ is summable over n. It is clear how to set up an electronic device that will compute an approximation of $\int_0^t X_{-s}(\omega)dZ_s(\omega)$ as it receives the signals $X_s(\omega)$ and $Z_s(\omega)$.

Here is a sketch of the proof. Since X has no oscillatory discontinuities and is bounded on every bounded interval a.s., the T_i^n increase without bound as $i \longrightarrow \infty$. Consider the left continuous process

$$X^n = \sum_i X_{T_i^n} \cdot 1_{((T_i^n, T_{i+1}^n]]}.$$

Clearly $|X^n - X| \leqslant 2^{-n}$ uniformly on $B = \Omega \times [0, \infty)$. Now $Y_t^n = \int_0^t X^n\, dZ$, and by the Dominated Convergence Theorem, $Y_t^n \longrightarrow \int_0^t X\, dZ$ for all $t > 0$, *in measure.* This is not yet good enough,but it reduces everything to showing (1). To do this, choose $q = 2$ and let P' denote the measure provided by the lemma. We can arrange matters so that $|X|_T^* \in L^2(P')$ as well. This implies that $X^n \cdot 1_{((0,T]]}$ belongs to $L_1[Z, P']$. Using in the third line below a semimartingale analogue of an inequality of Burkholder-Davis-Gundy [4] we obtain

$$\begin{aligned}\int \sum |Y^{n+1} - Y^n|_T^*\, dP' &= \sum \rho_1(|Y^{n+1} - Y^n|_T^*; P') \\ &= \sum \rho_1(|(X^{n+1} - X^n) * Z|_T^*; P') \\ &\leqslant \sum C_1 \cdot G_1(1_{((0,T]]}; (X^{n+1} - X^n) * Z, P') \\ &= C_1 \sum G_1((X^{n+1} - X^n) \cdot 1_{((0,T]]}; Z, P') \\ &\leqslant C_1 \sum G_1(2^{-n+1} \cdot 1_{((0,T]]}; Z, P') \\ &= C_1 \sum 2^{-n+1} G_1(1_{((0,T]]}; Z, P') < \infty.\end{aligned}$$

Hence $\Sigma |Y^{n+1} - Y^n|_T^* < \infty$ P-a.s. Here C_1 is a universal constant. The passage to P' is needed so the upper gauge used is homogeneous and the factors 2^{-n+1} can be taken out in the equality of the last line.

AN APPLICATION. Let F be a Lipschitz function. For $x \in \mathbf{R}$ set $X^{x,0} = x$ and $X^{x,n+1} = x + F(X_-^{x,n}) * Z$. Then $X^{x,n}$ converges uniformly on bounded intervals to the unique solution of $X^x = x + F(X_-^x) * Z$. In fact, $\Sigma |X^{x,n+1} - X^{x,n}|_T^* < \infty$ a.s. for all a.s. finite stopping times T [3], [8]. The r.c.l.l. iterates are only defined up to indistinguishability each. However, they can be fixed further as follows: when the algorithm provided by the theorem and used to compute $F(X_-^{x,n}) * Z$ converges set $X^{x,n+1}$ equal to its limit; else set $X^{x,n+1} = 0$. Similarly, if $\Sigma |X^{x,n+1} - X^{x,n}|_t^* < \infty$ for all $t > 0$ set $X^x = \lim X^{x,n}$; else set $X^x = 0$. Evidently $X_t^x(\omega)$ is arrived at by measurable operations from x and $(Z_s(\omega)\colon\ 0 \leqslant s \leqslant t)$. It is a version that depends in a jointly $\mathcal{B}(\mathbf{R}) \times \mathcal{F}_t$-*measurable way on* (x, ω).

Essentially the same remark applies when $Z = \vec{Z}$ is a vector of semimartingales and F a matrix of local Lipschitz functions. It shortens considerably the proof [8] that if Z is a (strong) Markoff process then so is the pair $(\vec{X}, \vec{Z})$. It obviates the need of finding a Doob-decomposition $\vec{Z} = \vec{M} + \vec{V}$ common to each starting probability of $\vec{Z}$.

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ADDED IN PROOF. I was advised by Professor Dellacherie that the identity of L^0-integrators with semimartingales was known to him and G. Mokobodzki for over a year and is to be published in *Seminaire de Probabilités de Strasbourg* XIII, Lecture Notes in Math., Springer-Verlag, Berlin. Their proof, simplified by G. Letta, amounts in essence to establishing the factorization theorem of Maurey-Rosenthal for the case $q = 1$. Note that at present factorization through L^2 is required in the proof of Theorem 2 and similar results [3].

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