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ON THE DIFFERENCE BETWEEN CONSECUTIVE PRIMES

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It was shown by Huxley [1] that

$$\pi(x) - \pi(x-y) \sim \frac{y}{\log x} \qquad (x^{\vartheta} \leqslant y \leqslant \tfrac{1}{2}x), \tag{1}$$

for any constant $\vartheta > 7/12$. It follows that

$$p_{n+1} - p_n \ll p_n^{\vartheta} \tag{2}$$

for $\vartheta > 7/12$, where p_n is the nth prime number. At present the asymptotic formula (1) is not known for any $\vartheta \leqslant 7/12$. However Iwaniec and Jutila [2] have recently shown that, if one asks only for

$$\pi(x) - \pi(x-y) \gg \frac{y}{\log x} \qquad (x^{\vartheta} \leqslant y \leqslant \tfrac{1}{2}x), \tag{3}$$

then $\vartheta \geqslant 13/23$ is admissible. It follows that (2) holds with $\vartheta = 13/23$. Here $7/12 = 0\cdot 5833\ldots$, while $13/23 = 0\cdot 5652\ldots$. Moreover they indicated that the condition $\vartheta \geqslant 13/23$ could be relaxed to $\vartheta > 5/9 = 0\cdot 5555\ldots$, by an elaboration of the argument. The constant 5/9 was the limit of their method.

We can now extend the range of validity of (2) and (3) as follows.

THEOREM. *For any $\vartheta > 11/20$ and $x \geqslant x(\vartheta)$ we have*

$$\pi(x) - \pi(x-y) > \frac{1}{212}\frac{y}{\log x}$$

in the range $x^{\vartheta} \leqslant y \leqslant \tfrac{1}{2}x$. Thus

$$p_{n+1} - p_n \ll p_n^{\vartheta}.$$

Note that $11/20 = 0\cdot 5500\ldots$. This constant is the limit of the present method, since $\vartheta > 11/20$ is required in the lemma quoted below.

The proof of our theorem, like that given by Iwaniec and Jutila, uses a combination of the linear sieve and certain weighted zero-density estimates for

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the Riemann zeta-function. However, we use a sharper bound (see below) for the remainder term in the linear sieve. Moreover, we take account of several positive contributions in our basic decomposition of $\pi(x) - \pi(x-y)$ in (4), which were ignored by Iwaniec and Jutila.

We now give an outline of the proof. First we must introduce some notation from sieve theory. For any finite set A of integers we define

$$A_d = \{n \in A; d \mid n\},$$
$$S(A, z) = \#\{n \in A; p \mid n \Rightarrow p \geqslant z\},$$
$$W^-(A, z, D) = S(A, z) - \sum_{(D/p)^{1/3} \leqslant q < p < z} S(A_{qp}, q),$$

where q and p run over primes, and $2 \leqslant D \leqslant z^4$. We then have the fundamental Buchstab identity, namely

$$S(A, z_2) = S(A, z_1) - \sum_{z_1 \leqslant p < z_2} S(A_p, p).$$

We take

$$A = \{n; x - y < n \leqslant x\},$$

then

$$\begin{aligned}
\pi(x) - \pi(x-y) &= S(A, x^{1/2}) \\
&= S(A, z) - \sum_{z \leqslant p < D^{1/2}} S(A_p, p) - \sum_{D^{1/2} \leqslant p < x^{1/2}} S(A_p, p) \\
&= W^-(A, z, D) + \sum_{(D/p)^{1/3} \leqslant q < p < z} S(A_{qp}, q) - \sum_{z \leqslant p < D^{1/2}} S(A_p, p) \\
&\quad - \sum_{D^{1/2} \leqslant p < x^{1/2}} S(A_p, (D/p)^{1/3}) + \sum_{\substack{D^{1/2} \leqslant p < x^{1/2} \\ (D/p)^{1/3} \leqslant q < p}} S(A_{qp}, q) \\
&= \Sigma_1 + \Sigma_2 - \Sigma_3 - \Sigma_4 + \Sigma_5, \qquad (4)
\end{aligned}$$

say. We give a lower bound for Σ_1 by means of the linear sieve. Usually the sieve is applied to give bounds for $S(A, z)$, and these involve a parameter D. However the *same* lower bound applies to the smaller quantity $W^-(A, z, D)$. This saves a term Σ_2. To apply the sieve we need an estimate for a remainder sum and this is provided by the following lemma.

LEMMA. *Let* $\eta > 0$, $11/20 + 2\eta < \vartheta < 7/12$, $0 \leqslant \phi \leqslant (6\vartheta - 1)/5 - 3\eta$ *and* $1 \leqslant M, N < x^{\vartheta}$. *Suppose* $|a_m|, |b_n| \leqslant 1$. *Then*

$$\sum_{\substack{M<m\leqslant 2M \\ N<n\leqslant 2N}} a_m b_n \left(\left[\frac{x}{mn} \right] - \left[\frac{x-y}{mn} \right] - \frac{y}{mn} \right) \ll x^{\vartheta-\delta}$$

for some $\delta = \delta(\eta) > 0$.

This is a crucial improvement over the corresponding result of [2].

The term Σ_4 in (4) is also estimated by the linear sieve, from above. However because of the summation over primes occurring in Σ_4, there will be a corresponding sum over primes in the remainder term. The range of this sum is too large to be dealt with by a direct appeal to the above lemma, and we therefore apply Vaughan's identity, which enables us to split the range into manageable parts.

For Σ_3 we give an asymptotic formula, by using weighted zero-density estimates. We also apply such estimates to Σ_2 and Σ_5. However only certain subranges of p and q can be dealt with in this way, and the remaining terms, being nonnegative, are discarded. Of course, to discard judiciously chosen nonnegative terms is the underlying idea in any combinatorial sieve method.

Full details of the proof will be published elsewhere.

REFERENCES

1. M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164–170.

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