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Topics in group rings, by Sudarshan K. Sehgal, Monographs and Textbooks in Pure and Applied Mathematics, vol. 50, Marcel Dekker, New York and Basel, 1978, vi + 251 pp., $24.50.

The theory of the group ring has a peculiar history. In some sense, it goes back to the 1890's, but it has emerged as a separate focus of study only in relatively recent times. We start with the early development of the theory of representations of finite groups over the complex field. Most people familiar with this think immediately of Frobenius and Burnside, who used approaches that seem unsuitable and even bizarre in the light of modern treatments. Admittedly, Frobenius' group determinant and Burnside's Lie-theoretic approach both yielded the basic properties of complex *characters*. However, they said much less about the *representations* themselves. For this reason, they have little application to the important problems of finding properties of representations over other rings–representations over fields of finite characteristic and over rings of algebraic integers have very important applications in group theory, algebraic number theory and topology. Hence, a more flexible approach was needed. In fact, the groundwork was being done by the little-known Estonian innovator Theodor Molien. Molien developed a theory of algebras over the complex field that included many of the features of the more general theory later developed by Wedderburn. He applied his theory to the case of representations of groups as follows: the Cayley representation of a finite group G as a permutation group on itself can be linearized to obtain a faithful representation of G in $\mathrm{GL}(n, \mathbf{C})$, where n is the order of G. The linear span of the image of this representation is a subalgebra of the algebra of $n \times n$ matrices. When it is analyzed by Molien's methods, the basic properties of the irreducible characters are deduced from properties of the irreducible representations.

This useful point of view lay dormant until the late 1920's, when Emmy Noether considered group representations as an illustration of her results on rings and modules. Her approach went to a new level of abstraction. Instead of the enveloping algebra of the regular representation, the group ring as it is now defined appeared. We may as well define it in full generality: let R be any associative ring, G any group (not necessarily finite). The group ring RG is the free R-module on the elements of G, with multiplication induced by that of G. That is, RG consists of formal sums $\Sigma_{g \in G} a_g G$, with all but finitely many $a_g = 0$ subject to

(i) $\Sigma a_g g = \Sigma b_g g$ if and only if $a_g = b_g$ for all g.
(ii) $\Sigma a_g g + \Sigma b_g g = \Sigma (a_g + b_g)g$.
(iii) $\Sigma (a_g g)(\Sigma b_h h) = \Sigma c_t t$, where $c_t = \Sigma_{gh=t} a_g b_h$.

For those accustomed to the view of group algebras adopted by functional analysts, one can consider RG as the set of functions from G to R which are zero at all but finitely many places, with pointwise addition and convolution as multiplication. Once this definition has been made, the representations of G on R-modules can be viewed as (RG, R)-bimodules (plain left RG-modules suffice when R is commutative, which is the case almost always considered).

Thus, we have a mechanism that facilitates representation theory. However, we get a great bonus. We now have a class of rings in which calculation is relatively easy. So, we can ask how the ring-theoretic properties of RG are influenced by those of R and by the group-theoretic properties of G, and conversely. Also, we have a proving ground for conjectures about rings. It is this bonus that has been exploited in recent years.

Let us survey briefly some of the topics that have been studied in this area. For more complete information, the reader should consult D. S. Passman's book *Algebraic structure of group rings*, Interscience, New York, 1977 or his survey paper, *Advances in group rings*, Israel J. Math. **19** (1974), 67–107. Perhaps the problem of greatest notoriety in the theory is the semisimplicity problem. Thus, let $J(RG)$ denote the Jacobson radical of RG (i.e. the intersection of its maximal left ideals). The problem is to tell when $J(RG) = 0$. The famous theorem of Maschke handles the case of KG, with K a field and G finite. If the characteristic of K is zero, KG is semisimple for all finite G. If K has positive characteristic p, then $J(KG) = 0$ if and only if p does not divide the order of G. For infinite groups, the problem is open, although many cases have been attacked successfully. The best known result was proved by Amitsur in 1959: if K has characteristic zero and is not algebraic over the rationals, then KG is semisimple, for all groups G. Passman and Chalabi obtained an analogous result in finite characteristic: if K has characteristic p but is not algebraic over the integers modulo p, and if G has no elements of order p, then $J(KG) = 0$. There are also some special classes of groups for which the problem has been solved. Hampton, Passman and Zalesskiĭ solved the problem when K is of finite characteristic p and G is solvable. Also in characteristic p, Zalesskiĭ and Passman solved the problem for linear groups also in characteristic p.

A related problem is that of primitivity. A ring is *left primitive* if it has a faithful, irreducible left module. For some time, it was not known whether any primitive group algebras existed. In 1972, Formanek and Snider showed that if G is locally finite and countable, then KG is primitive if and only if $J(KG) = 0$ and every nonidentity element of G has infinitely many distinct conjugates. Hence, if G is the "infinite symmetric group" (the group of those permutations of a countable set that leave all but finitely many elements fixed), and K is, say, the complex field, then KG is primitive. Perhaps more surprisingly, they showed that for any KG, there is a group H containing G such that KH is primitive. The following year, Formanek showed that if A and B are nontrivial groups, not both of order two, then RG is primitive, where G is the free product $A * B$ and R is a ring without zero divisors, whose cardinality does not exceed that of G. By the way, this sensitivity to the cardinality of the coefficient ring is a real concern. For, it can be shown that if $G = \mathbf{Z} \times (\mathbf{Z} * \mathbf{Z})$ ($\mathbf{Z}$ the additive group of integers) and K is a field, then KG is primitive if and only if K is countable.

There are, of course, many other classes of rings defined by ideal-theoretic properties. Many of these have been investigated for group algebras. For example, Connell, in his remarkable paper *On the group ring*, Canad. J. Math. **15** (1963), 650–685, showed that KG is prime (meaning that the product of any two nonzero ideals is nonzero) if and only if $\Delta(G) = \{g \in G \mid g$ has only finitely many conjugates$\}$ is torsionfree abelian. Passman analogously characterized those group rings that are semiprime (i.e. subdirect products of prime rings).

In other directions, one can look at properties of single elements. In any ring, it is always interesting to know the idempotent elements (those that are their own squares) and the nilpotent elements (those having a power equal to zero). Passman and Connell obtained necessary conditions for elements to be nilpotent, and consequently, sufficient conditions for the group ring to have no nonzero nil ideals (ideals consisting entirely of nilpotent elements). About idempotents, there are some very amusing results. Kaplansky showed in 1969 that if K has characteristic zero, and $0 \neq e \in KG$ is idempotent, then $0 < \operatorname{tr}(e) \leqslant 1$ and $\operatorname{tr}(e) = 1$ if and only if $e = 1$. Here, $\operatorname{tr}(e)$ is simply the coefficient of the identity element of G in e. This has the consequence that KG is von Neumann finite–if $\alpha\beta = 1$ in KG, then $\beta\alpha = 1$. In 1972, Zalesskiĭ showed that $\operatorname{tr}(e)$ is actually a rational number, in the case considered by Kaplansky, and that it is in $GF(p)$ if K has characteristic p. Along more global lines, Formanek showed in 1973 that if K has characteristic 0 and if $x \in G$ can be conjugate to x^n only when $n = \pm 1$, then KG has no nontrivial idempotents.

There has also been interest in the problem of group algebras that satisfy polynomial identities. If R is an algebra over a field K, we say that R satisfies a polynomial identity of degree n if there is a polynomial $f(X_1, \ldots, X_k)$ of degree n over K, in noncommuting variables, such that $f(a_1, \ldots, a_k) = 0$ for all $a_1, \ldots, a_k \in R$. For instance, a commutative K-algebra satisfies $X_1X_2 - X_2X_1 = 0$; indeed, satisfaction of such identity is regarded as a suitable generalization of commutativity by many ring theorists. The basic result on polynomial identities, proved by Isaacs and Passman in 1964, asserts that if K has characteristic zero and KG satisfies a polynomial identity of degree n, then G has an abelian subgroup A whose index in G is bounded by a function of n alone. Passman further showed that $[G: \Delta(G)] \leqslant n/2$ and that $\Delta(G)$ has finite commutator subgroup. These matters are also connected with bounds on the dimensions of the irreducible KG-modules.

One could go on at great length citing further results of great interest. However, the specialists know them, and the nonspecialists are probably close to shell-shock, so let us remark on only one more thing. It is clearly important to know whether the group ring determines the group and the ring. There has been much effort spent in considering this (slightly weaker) form of the problem: if RG and RH are isomorphic, are G and H isomorphic? The answer is clearly negative in general. For example, if G and H are finite abelian groups of the same order, then $\mathbf{C}G$ and $\mathbf{C}H$ are isomorphic. However, their rational group algebras are distinct, as was shown by Perlis and Walker in 1950. So, there remains the hope that distinct groups have distinct group rings over some reasonable coefficient ring. One might even have hoped that a coefficient field could be used. Unfortunately, Dade threw a considerable quantity of cold water on this dream in 1971, when he produced two nonisomorphic finite groups that have isomorphic group algebras over *every* field. However, there remains the outstanding problem: does $\mathbf{Z}G \cong \mathbf{Z}H$ imply $G \cong H$? The answer is known to be affirmative for many classes of groups.

Sehgal has written an interesting and useful monograph. It is not, and does not claim to be, a comprehensive volume. Rather, he has collected many results on the research topics pursued by him and his school. As such, it should be viewed as a supplement to the book of Passman mentioned above. However, it has a separate identity, since it deals extensively with coefficient rings that are not fields, an area that Passman seldom mentions.

The book starts with a discussion of idempotents. In addition to the basic results of Zalesskiĭ and Passman cited above, there is a discussion of the refinements due to Bass, Cliff and Sehgal, giving deeper information about the rational number $\operatorname{tr}(e)$ and related quantities. Also, there are discussions of when RG has nontrivial idempotents, and of the coefficients of central idempotents. Next comes one of two (separated) chapters on units. Recall that a unit of a ring R is an element that has a two-sided multiplicative inverse, and that the set of all such forms a group $U(R)$. In the group ring case, $U(RG)$ contains a subgroup isomorphic to G, and one can ask myriad questions about the relationship between these groups. Generally, one wants to know how much bigger $U(RG)$ is than G. So, one searches for so-called nontrivial units, i.e. those not of the form rg, with $r \in U(R)$ and $g \in G$. One can also inquire as to when any given group-theoretic property appears in $U(RG)$. One old but satisfying theorem in this area was supplied by G. Higman in 1940: if G is a torsion group, then $U(\mathbf{Z}G) = \{\pm g | g \in G\}$ if and only if G is either abelian of exponent four or six, or a Hamiltonian 2-group. Sehgal provides a complete proof of this theorem. Further, he shows that these are exactly the conditions under which $U(\mathbf{Z}G)$ is periodic.

The other chapter on units (which appears as Chapter VI) deals with group theoretic properties of $U(RG)$. After a discussion of some rather technical problems, there is a complete presentation of the characterization by Seghal and Zassenhaus of those groups G for which $U(\mathbf{Z}G)$ is nilpotent. Also presented are Bateman's conditions for $U(KG)$ to be solvable (which apply when G is finite and K is a field), some results on when $U(RG)$ is an

FC-group (one in which the conjugacy classes are finite), and on normal subgroups of $U(\mathbf{Z}/n\mathbf{Z}[G])$.

The third chapter of the book concerns the isomorphism problem discussed earlier, i.e. the problem of determining when $RG \cong RH$ implies $G \cong H$. Many basic results are discussed, but Dade's important example is unfortunately omitted. Chapter IV deals with the related problem of uniqueness of the coefficient ring: does $RG \cong SG$ imply $R \cong S$? As for the isomorphism problem, it is easy to see that the answer is negative in general, but one may still search for special conditions under which it becomes affirmative. Since little is known about this problem, the author confines his attention to the case where $G = \langle x \rangle$ is infinite cyclic. Hence, RG in this case is the ring $R\langle x \rangle = R[x, x^{-1}]$ of Laurent polynomials over R. For this special context, some results of Sehgal and Parmenter are presented, which show the answer to the uniqueness problem to be affirmative for some special classes of rings (perfect, commutative von Neumann regular, commutative local and a few others).

Further, there is a chapter on Lie properties of KG. Here, KG is viewed as a Lie algebra by the usual device of defining $[a, b] = ab - ba$. One may then ask for conditions that KG be solvable, nilpotent or whatever, when viewed as a Lie algebra in this fashion. As a sample, we cite the theorem of Passi, Passman and Sehgal: let K have characteristic $p \geqslant 0$. Then KG is Lie solvable if and only if G is p-abelian, if $p \neq 2$ or $p = 2$ and G has a 2-abelian subgroup of index at most two. (Here, G is called p-abelian if its derived group is a finite p-group; 0-abelian if it is abelian in the usual sense.)

The book concludes with a compilation of research problems which were stated at various points in the text. Forty-two such problems (of clearly variable difficulty) are nicely organized, with comments showing the connections between them. This chapter should be very useful for researchers in the field (especially beginners), and the author is to be congratulated for providing it.

The entire book is well written and carefully organized. It is inherent in the material that some of the proofs are computational and somewhat boring, but the dilettante can easily skip over the tedious parts, and follow the flow of ideas. The format and typography are uninspired, but straightforward enough to be undistracting. In all, this book is quite pleasing, as specialized works go, and I think that anyone with any interest in group rings will find some valuable nuggets in it.

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K-theory, an introduction, by Max Karoubi, Springer-Verlag, Berlin, Heidelberg, New York, 1978, xviii + 308 pp., $39.00.

What is a real vector space of dimension -2? What is an abelian group of order $1/3$? Assuming that the reviewer retains some measure of sanity, which